Math 141: Calculus II

Final Exam Revision Guide

Exam Date: Tuesday, December 11th 14h00

1 Things we should've remembered from Calculus I....

Derivatives:

1. $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}$, **2.** $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$, **3.** $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$,

4. $\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}$, **5.** $\dfrac{d(u/v)}{dx} = \dfrac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, **6.** $\dfrac{d(e^{cu})}{dx} = ce^{cu}\dfrac{du}{dx}$,

7. $\dfrac{d(c^u)}{dx} = (\ln c)c^u\dfrac{du}{dx}$, **8.** $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$,

9. $\dfrac{d(\sin u)}{dx} = \cos u\dfrac{du}{dx}$, **10.** $\dfrac{d(\cos u)}{dx} = -\sin u\dfrac{du}{dx}$,

11. $\dfrac{d(\tan u)}{dx} = \sec^2 u\dfrac{du}{dx}$, **12.** $\dfrac{d(\cot u)}{dx} = \csc^2 u\dfrac{du}{dx}$,

13. $\dfrac{d(\sec u)}{dx} = \tan u \sec u\dfrac{du}{dx}$, **14.** $\dfrac{d(\csc u)}{dx} = -\cot u \csc u\dfrac{du}{dx}$,

15. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, **16.** $\dfrac{d(\arccos u)}{dx} = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}$,

17. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$, **18.** $\dfrac{d(\operatorname{arccot} u)}{dx} = \dfrac{-1}{1+u^2}\dfrac{du}{dx}$,

19. $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **20.** $\dfrac{d(\operatorname{arccsc} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$,

5 Integrals

Main idea: you can approximate areas under a curve using rectangles. As the number of rectangles increases, the approximation becomes more accurate. So, if you take the limit as the number of rectangles tend to infinity, you obtain a precise measurement of the area bounded by a curve. This infinite limit can be expressed as an integral.

5.1 Areas and Distances

Definition 5.1. Area $\rightarrow$ the area of the region S lying under a continuous function f is the limit of the sum of the areas of the approximating rectangles.

$$A = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + ... + f(x_i)\Delta x]$$

5.2 The Definite Integral

Definition 5.2. Definite integral $\rightarrow$ if f is a function defined for $a \geq x \geq b$, we sub-divide the interval [a,b] into n sub-intervals of equal width:

- $\Delta x = \frac{b-a}{n}$
- $a = x_0$
- $b = x_n$
- The **Definite integral of f from a to b is:**

$$\int_a^b f(x)dx = \lim_{n \Rightarrow \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{1}$$

- The integral measures the area captured between the function, $f(x)$, and the x-axis.
- For areas above the x-axis, this is registered as positive.
- For areas under the x-axis, this is registered as negative.
- So, the integral measures the **net area** of a function.

Theorem 1 (Riemann Sum And Integral Connection)**.** *The connection between integrals and Riemann sums is as follows:*

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x)\Delta x \tag{2}$$

Where

$$\Delta x = \frac{b-a}{n}$$
$$x_i = a + i\Delta x$$

Rules for evaluating sums (summation rules):

Power of i	**Sum formula**
$\sum_{i=1}^{n} i$	$\frac{n(n+1)}{2}$
$\sum_{i=1}^{n} i^2$	$\frac{n(n+1)(2n+1)}{6}$

5.3 Fundamental Theorem of Calculus

Motivation: establishes a connection between the two branches of calculus: differential calculus and integral calculus. The main idea:

- Differentiation and integration are **inverse processes**.
- There are two parts to the theorem.

Theorem 2 (Fundamental Theorem of Calculus Part 1)**.** *(1) If we first integrate f then differentiate the result, we return to the original function f. (2) If f is continuous, then the integral can be interpreted as the "area so far" function:*

$$g(x) = \int_a^x f(t)dt, \; x \in [a, b] \tag{3}$$

Theorem 3 (Fundamental Theorem of Calculus Part 2)**.** *If f is continuous on [a,b], then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

where f(x) is any anti-derivative of F(x).

Remark: if the limits of integration are expressed in terms of <u>functions</u>, then we obtain the following:

$$\int_{\mathbf{a(x)}}^{\mathbf{b(x)}} \mathbf{f(x)dx} = \mathbf{a'(x)f(a(x)) - b'(x)f(b(x))} \tag{4}$$

5.4 Indefinite Integrals and the Net Change Theorem

Integrals to **memorize**! Do not forget the $+\mathbf{C}$ constant.

Table of Integration Formulas Constants of integration have been omitted.

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$

3. $\displaystyle\int e^x\,dx = e^x$

4. $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b}$

5. $\displaystyle\int \sin x\,dx = -\cos x$

6. $\displaystyle\int \cos x\,dx = \sin x$

7. $\displaystyle\int \sec^2 x\,dx = \tan x$

8. $\displaystyle\int \csc^2 x\,dx = -\cot x$

9. $\displaystyle\int \sec x \tan x\,dx = \sec x$

10. $\displaystyle\int \csc x \cot x\,dx = -\csc x$

11. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$

12. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x|$

13. $\displaystyle\int \tan x\,dx = \ln|\sec x|$

14. $\displaystyle\int \cot x\,dx = \ln|\sin x|$

15. $\displaystyle\int \sinh x\,dx = \cosh x$

16. $\displaystyle\int \cosh x\,dx = \sinh x$

17. $\displaystyle\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

18. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$

***19.** $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

***20.** $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Theorem 4 (Net Change Theorem)**.** *The integral of a rate of change is the net change. This is because:*

$$\int_a^b F'(x)dx = f(b) - f(a)$$

Remarks

1. If you are asked to compute the total area under a curve, then you need to integrate the absolute value of the function to avoid subtracting "negative area."

5.5 Substitution Rule

Motivation: if the integral is complicated, then you can substitute in other variables to turn a relatively complicated integral into a simpler integral.

1. We can accomplish this by changing from the original variable x to a new variable, u, that is a function of x.
2. If you are given a definite integral, then it is useful to **change the limits of integration** whenever you make a substitution.

The Substitution Rule: if $u = g(x)$ is a differentiable function whose range is an interval I and is continuous on that interval I, then

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du \tag{5}$$

Note that the substitution rule is the "inverse" of the chain rule from Calculus I.

5.5.1 Symmetry

Integrating symmetric integrals can be really simple:

- If f is even (that is, $f(-x) = f(x)$, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \tag{6}$$

- If f is odd (that is, $f(-x) = -f(x)$, then

$$\int_{-a}^{a} f(x)dx = 0 \tag{7}$$

6 Applications of Integration

6.1 Area Between Curves

Motivation: by taking differences of areas under curves, we can compute the area between two curves. This is useful for determining volume later on.

Definition 6.1. Area $\rightarrow$ the area A of a region S bounded by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is given by

$$A = \int_{a}^{b} (|f(x) - g(x)|)dx \tag{8}$$

This formula is doing nothing more than taking the area between the upper curve and the x-axis, and subtracting the area between the lower-curve and the x-axis.

Remark 6.1. If it easier to integrate the inverse of the functions that you are given (that is, if the functions are easier expressed as a function of y, then the formula is basically the same, except the variable of integration is different:

$$A = \int_{x}^{d} (|f(y) - g(y)|)dy$$

6.2 Volumes

Motivation: using the idea that we can compute the area between two curves, the volume of a region formed by revolving lines about certain axes or lines can be obtained by treating each cross-sectional area formed as a sort of "Riemann sum", and taking the limit as the number of cross-sectional areas tend to infinity. Visual representation:

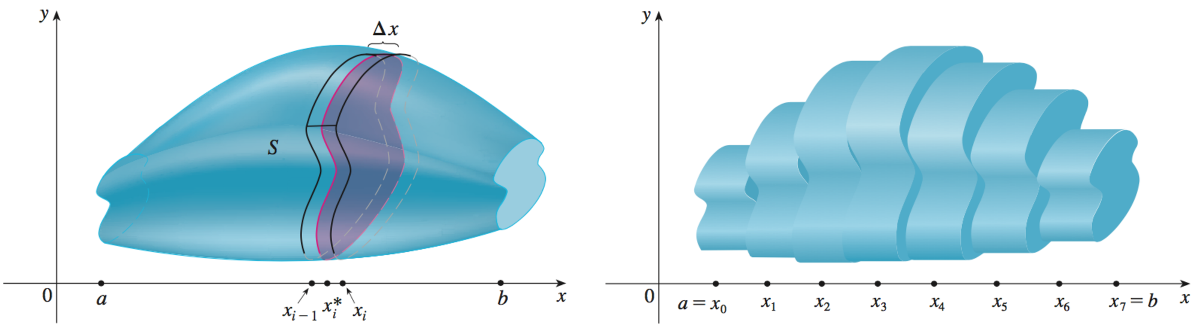


FIGURE 3

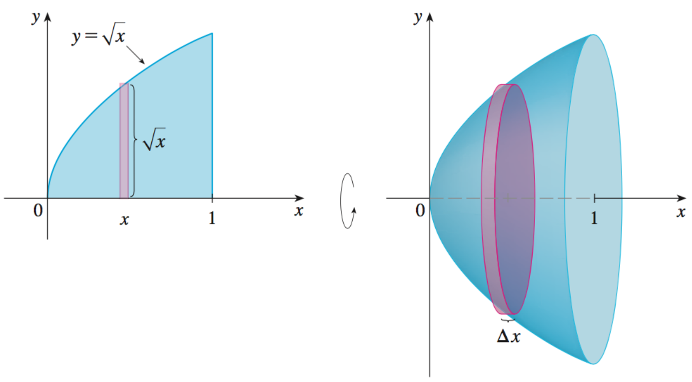
Then, volume is nothing else than the integral of the area from the lower bound of the object to the upper bound of the object:

$$V(x) = \int_a^b A(x)dx$$

How do we obtain $A(x)$?
There are two general methods: **cylindrical shells** and the **washer method**.

6.2.1 Washer Method

The washer method computes areas by determining **solids of revolution**, which are solids obtained by revolving a region about a line. Visually, it looks like this:



(a) (b)

If the cross-section is a **disc** (no hole in the middle), then the volume is given by:

$$V(x) = \int_a^b [\pi(r^2)]dx \tag{9}$$

Where r is given by the value $f(x)$ if the area is in terms of x or $f(y)$ if the area is defined in terms of y.

If the cross-section is a **washer**, then the volume is given by:

$$V(x) = \int_a^b (\pi[r_{outer}^2 - r_{inner}^2])dx \tag{10}$$

6.3 Volumes by Cylindrical Shells

Motivation: we use this method when it is easier to obtain the volume of a solid by rotating about the y-axis. Instead of summing infinite discs or washers, cylindrical shells sums up infinite cylinders. **Main idea**: this method approximates area with a bunch of cylinders by taking the limit as the number of cylinders tends to infinity. Visually, it look like this:

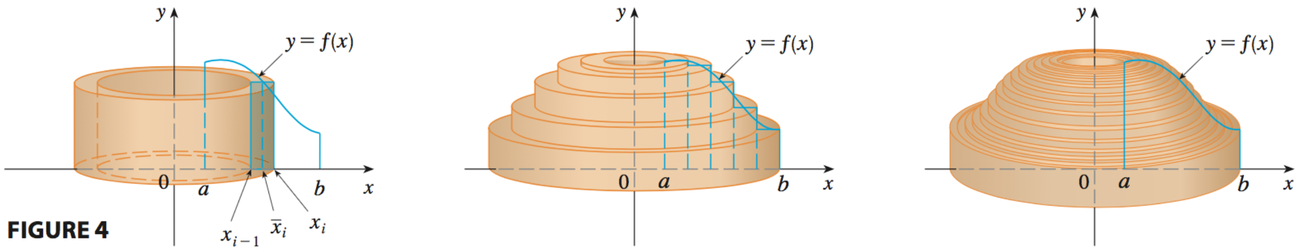


FIGURE 4

The area would then be given by:

$$V(x) = \int_a^b 2\pi x f(x) dx \tag{11}$$

Intuitively, this means

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \quad \underbrace{[f(x)]}_{\text{height}} \quad \underbrace{dx}_{\text{thickness}}$$

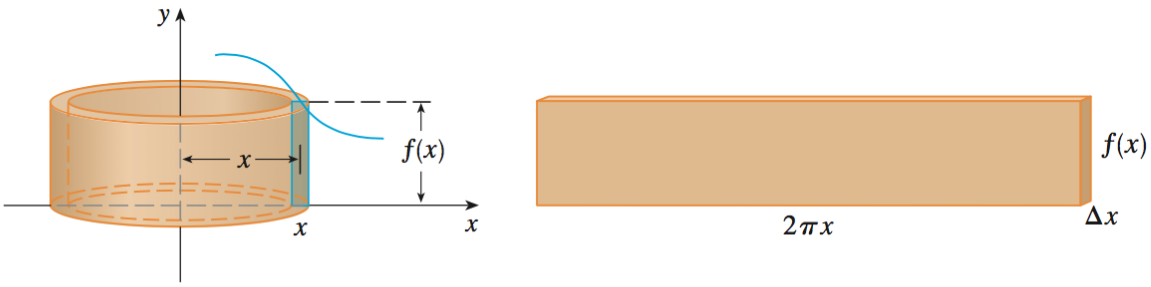


FIGURE 5

6.4 Average Value of a Function

Motivation: we can actually use integrals to calculate the average value of infinitely many observations.

Definition 6.2. Average value → the average value of f on the interval [a,b] is given by

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx \tag{12}$$

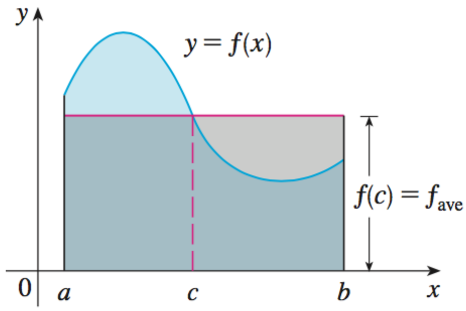
Something that's closely related to this is the **mean value theorem for integrals**. It is:

Definition 6.3. Mean Value Theorem for Integrals → if f is continuous on [a,b], then there exists a number c in [a,b] such that:

$$f(c) = f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx \text{ or, alternatively,} \tag{13}$$

$$\int_a^b f(x) dx = f(c)(b-a) \tag{14}$$

Visually, this means:



- For a positive function f, there is a number c such that the rectangle with base [a,b] and height $f(c)$ has the same area as the region under the graph f from a to b.

7 Techniques of Integration

7.1 Integration by Parts

Motivation: each differentiation rule has a corresponding integration rule:

1. Substitution rule corresponds to the chain rule.
2. Integration by parts corresponds to the product rule.

The integration by parts formula is given by:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \tag{15}$$

Some tricks to remember when using this technique

- Use **cyclic integration** when you see that you have e^x and sine or cosine.
- Sometimes, you need to use multiple iterations of the rule to obtain something useful/easy to integrate.
- Choose $f(x)$ to be a function that does not get more complicated when you take the derivative and $g(x)$ to be a function that is easy to integrate. In general, this usually means that you set $f(x)$ to be the polynomial if there is one in the integral.

The formula to evaluate a definite integral using integration by parts is given by:

$$\int_a^b f(x)g'(x)dx = \Big[f(x)g(x)\Big] - \int_a^b f'(x)g(x)dx \tag{16}$$

7.2 Trigonometric Integrals

Motivation: two strategies will be introduced: one for evaluating $\int \sin^m(x)cos^n(x)dx$ and one for evaluating $\int \tan^m(x)sec^n(x)dx$. Here, trig identities are absolutely integral to solving the problems, so here's what's essential to know:

$$\sin^2(x) = \frac{1}{2}(1-\cos(2x)) \qquad \text{(Half-Angle)} \tag{17}$$

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x)) \qquad \text{(Half-Angle)} \tag{18}$$

$$\sin^2(x)+\cos^2(x) = 1 \qquad \text{(Pythagorean - also know the alternate forms)} \tag{19}$$

$$\sec^2(x) = 1+\tan^2(x) \qquad \text{(alt. form of pythagorean)} \tag{20}$$

$$\sin^2(x) = 2\sin(x)\cos(x) \qquad \text{(Double-angle)} \tag{21}$$

Product Identities

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$
$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

Case 1: different powers of sine and cosine

Case 1.1: the power of cosine is odd
Then: save 1 $\cos(x)$ factor to obtain an even power of cosine. Use the pythagorean identity to express the remaining terms in terms of $\sin(x)$.

Case 1.2: the power of sine is odd
Then: save 1 factor of $\sin(x)$ to obtain an even power of sine. Then, use the pythagorean identity to express the remaining terms in terms of $\cos(x)$.

Case 1.3: the powers of sine and cosine both are even
Then: use the half-angle identities or the double angle identity to simplify the integral. Then, attack the problem with u-substitution or another known strategy.

Case 2: different powers of tan and sec

Case 2.1: the power of secant is even
Then: save a factor of $\sec^2(x)$ and use the pythagorean identity (alternative) to express the remaining factors in terms of $\tan(x)$.

Case 2.2: the power of tangent is odd
Then: save a factor of $\sec(x)\tan(x)$ and use the pythagorean identity (alternative) to express the remaining factors in terms of $\sec(x)$.

Case 2.3: else
Not so clear-cut here. Might have to use identities, integration by parts, etc. So, memorize the integrals of $\tan(x)$ and $\sec(x)$:

$$\int \tan(x)dx = ln(|\sec(x)|) + C \qquad \int \sec(x)dx = ln(|\sec(x) + \tan(x)|) + C \tag{22}$$

Case 3: trigonometric products
Integrals of this form: $\int \sin(mx)\cos(nx)dx$, $\int \sin(mx)\sin(nx)dx$, or $\int \cos(mx)\cos(nx)dx$. Then, use the product identities in the beginning of the section to simplify the integral, then integrate it like a regular integral.

7.3 Trigonometric Substitution

Motivation: "the worst part." We need to use this technique when we are dealing with finding areas of discs or ellipses, which are in the form of

$$\int_a^b \sqrt{a^2 - x^2}dx$$

We cannot integrate this using other methods, so we use **trigonometric substitution**. It's algorithmic:

1. Make a substitution in the form of $x = g(t)$. Need to restrict the domain of g to ensure that it's one-to-one (so we can obtain an inverse).
   (a) The goal is to obtain an integral of the form:
   $$\int f(x)dx = \int f(g(t))g'(t)dt$$
   (b) Make an appropriate substitution based on the expression under the square root. Precisely,

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

   (c) If the integral is definite: change the corresponding limits of integration to avoid using the triangle method.
2. Integrate the simplified expression. If it's definite, then evaluate the expression at the new limits of integration. Then, you're done. Else, move on.
3. **Triangle method**: draw a right triangle to determine the original value of x, and undo the substitution to obtain the original variable of integration, x.
4. Do not forget the $+C$.

7.4 Integration of Rational Functions by Partial Fractions

Motivation: finding a way to integrate a function that is expressed as a ratio of polynomials. This is done by breaking the fraction down so that it's the sum of simpler fractions.

The way that a polynomial ratio is divided into simpler ones is based on the nature of the denominator. Four cases:

1. The denominator is a product of distinct and linear factors, meaning that there are no repeated factors and that none of the factors are constant multiples of each other.
2. The denominator is a product of linear factors, some of which are repeated.
3. The denominator contains irreducible quadratic factors, none of which are repeated.
   - **Test for irreducibility**: when the following inequality is true:
   $$b^2 - 4ac < 0$$
4. The denominator contains a repeated, irreducible quadratic factor.

How to do it. Say we have the following polynomial
$$f(x) = \frac{P(x)}{Q(x)}$$

Then:

1. **Compare the degrees** of $P(x)$ and $Q(x)$. If the degree of $P(x)$ is LESS than $Q(x)$, we have a **proper fraction**: move on to step 2. Else
   (a) If the degree of $P(x)$ is greater than the degree of $Q(x)$, then a preliminary step must be carried out: use **long polynomial division** to divide $Q(x)$ into $P(x)$. We then obtain
   $$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$
   $S(x)$ is the quotient and $R(x)$ is the remainder. Move to step 2.
2. **Factor** as much as possible. Any polynomial can be factored as a product of linear factors in the form $(ax+b)$ and an irreducible factor (ax^2+bx+c).
   (a) By inspection: a root may stand out.
   (b) Trivial roots: $-2, -1, 0, 1, 2$.
   (c) Regular factoring techniques.
   (d) Long division by suspected factors if the degree is greater than 2.
3. **Sub-cases**. Express the proper function as the sum of partial fractions. Then, you need to determine which case we are in.
   (a) **Case 1)** $Q(x)$ is the factor of distinct, linear factors. Then, by the partial fractions theorem, there exists k constants $A_1, A_2, ..., A_k$ such that
   $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_k}{a_kx+b_k}$$
   Using partial fractions decomposition, find the constants.
   (b) **Case 2)** if $Q(x)$ is a factor of k linear factors with repetition. By the partial fractions theorem, there exists constants such that
   $$\frac{R(x)}{Q(x)} = \frac{A}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_n}{(a_1x+b_1)^n}$$
   Basically, every degree of n gets expressed if the linear factor is ever repeated.
   (c) **Case 3)** $Q(x)$ a distinct irreducible quadratic factor. We know this happens if the following holds:
   $$ax^2+bx+c, \text{ where } \Delta = b^2-4ac < 0$$
   By the partial fraction theorem, for irreducible quadratic terms, the following additional term will emerge:
   $$\frac{R(x)}{Q(x)} = \cdots + \frac{Ax+B}{ax^2+bx+c} + \cdots$$
   In this case, the following integration formula might be useful when trying to integrate an irreducible quadratic function:
   $$\int \frac{1}{x^2+a^2}dx = \frac{1}{\sqrt{a}} \arctan \frac{x}{\sqrt{a}} + C \tag{23}$$
   (d) **Case 4)** if $Q(x)$ contains irreducible quadratic factors, at least one of which is repeated. Then, we are in a case analogous to that of case 2. By the partial fractions theorem, there exists constants such that:
   $$\frac{R(x)}{Q(x)} = \cdots + \frac{A_1x+B_1}{Ax^2+bx+c} + \frac{A_2x+B_2}{(Ax+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax+bx+c)^n} + \cdots$$

4. **Integrate**. Most common integration/algebra methods that appear:
   (a) Complete the square
   (b) U-substitution
   (c) Using the integral of arctan.
   (d) Using the integral of x^{-1}.

Remark 7.1. If we have the following integral

$$\int \frac{x}{(1+x^2)^m} dx$$

Then, the following holds $\forall m$:

1. If $m = 1$, then

$$\int \frac{x}{(1+x^2)^m} dx = \frac{1}{2} ln(1+x^2) \tag{24}$$

2. If $m > 1$, then

$$\int \frac{x}{(1+x^2)^m} dx = \frac{1}{2^{m-1}} \times \frac{1}{(x^2+1)^{m-1}} \tag{25}$$

7.5 Integration techniques

Memorize all the integrals in the chart.

1. Simplify the integrand, if possible.
   (a) algebraically
   (b) trig identities
2. Look for an obvious substitution
3. Classify the integrand according to its form.

Table 1: Recognizing Integration Patters

Technique	**Hint**
Trigonometric integrals	If $f(x)$ is a product of powers of $\sin(x)$ and $\cos(x)$ or $\tan(x)$ and $\sec(x)$, then we use the substitutions recommended from that section.
Rational functions	if f is a rational function, then we use partial fractions.
Integration by parts	if $f(x)$ is the product of a power of x or a polynomial and a transcendental function (ex - trig, exponential, or log) , then use integration by parts
Radicals	certain forms of radial hint to certain subs $\sqrt{\pm x^2 \pm a^2} \Rightarrow$ inverse trig sub. $\sqrt[n]{ax+b}$ use the rationalizing substitution $u = \sqrt[n]{ax+b}$

4. Try again: there are really only two main integration methods: substitution and parts, so if it does not work the first time, then
   (a) Try substitution again
   (b) Integral by parts can sometimes work with single functions (like, integrating arctan).
   (c) Manipulate the integral again: rationalizing using the denominator and using trig identities.

7.6 Improper Integrals

Definition 7.1. Improper integral an improper integral is one where either the interval is infinite or if there exists an infinite discontinuity contained within the limits of integration.

7.6.1 Type I: infinite Integrals

Motivation: want to extend the concept of the definite integral to the case where the interval is infinite.

Definition 7.2. There are three cases of type I improper integrals

1. If $\int_a^t f(x)dx$ is finite for every number $f \geq a$, then we define the following integral:

$$\int_a^{+\infty} f(x)dx = \lim_{x\to+\infty} \int_a^t f(x)dx \tag{26}$$

2. If $\int_t^b f(x)dx$ exists, then for every number $t \leq b$, then we define the following integral:

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx \tag{27}$$

   For points 1) and 2), the following definitions apply:

   (a) **Convergent** $\rightarrow$ if the limit exists as a finite number, then we say that the infinite integral converges.

   (b) **Divergent** $\rightarrow$ if the corresponding limits do not exist as finite numbers.

3. If both $\int_a^t f(x)dx$ and $\int_t^b f(x)dx$ are convergent, then we can define the following:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{+\infty} f(x)dx \tag{28}$$

   where any $a \in \mathbb{R}$ can be used.

Remark 7.2. Why is $\frac{1}{x^2}$ convergent but $\frac{1}{x}$ is divergent? This is because the former is **sufficiently decreasing**, consequently producing a finite area; the other is not, consequently producing an infinite area. In fact, this is a rule:

$$\int_1^{+\infty} \frac{1}{x^p}dx \text{ is } \begin{cases} \text{convergent if } p > 1 \\ \text{divergent if } p \leq 1 \end{cases} \tag{29}$$

Remark 7.3. For case 3, a nice place to split an integral up is usually at 0 because it makes evaluating things easier.

7.6.2 Discontinuous Function

Motivation: you need to first check an integral for continuity; if there is a discontinuity, then you need to split the integral at the discontinuity.

Definition 7.3. 1. If f is a continuous function on $[a, b[$ and discontinuous at $x = b$, then we obtain:

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx \tag{30}$$

2. If f is continuous on $]a, b]$, and discontinuous at $x = a$, then:

$$\int_c^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx \tag{31}$$

(basically the same idea as point 1). The same sub-cases from the type I improper integral:

(a) **Convergent** if the corresponding limits exist as finite numbers.

(b) **Divergent** if the corresponding limits do not exist as finite numbers.

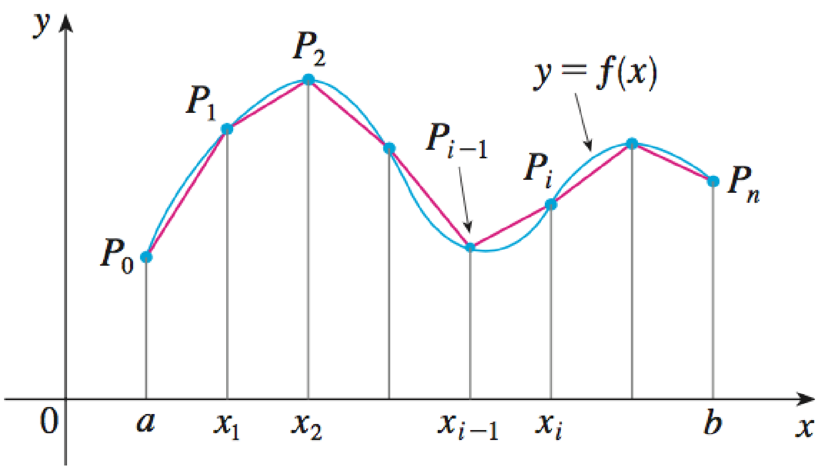
3. If f has a discontinuity at $c, c \in (a, b)$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{32}$$

8 Further Applications of Integration

8.1 Arc Length

Motivation: integrals can also be used to compute the length ℓ under the curve C. Main idea:



Definition 8.1. The arc length of the curve can be approximated, if we are given a function of x, with the following equation:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx \tag{33}$$

Definition 8.2. The idea to obtain the length of the curve defined in terms of y is identical:

$$L \int_c^d \sqrt{1 + [g'(y)]^2} dy \tag{34}$$

Definition 8.3. The **arc length function** is given by the following equation:

$$S(x) = \int_0^x \sqrt{1 + f'(t)^2} dt \tag{35}$$

It measures the arc length obtained so far: the length of the curve $y = f(x)$ from a particular starting point $P(a, f(a))$ to a point on the curve $Q(x, f(x))$.

8.2 Areas of Surface of Revolution

Motivation: to define this, you need a curve to rotate around an axis. We use the ideas from chapter 6 and section 8.1 to determine this.

Definition 8.4. A **surface of revolution** is formed when a curve is rotated around an axis. Such a surface is called the lateral boundary of the volume.

Definition 8.5. If we assume that f is positive and has a continuous derivative. Then, we define the surface of the area obtained by rotating the curve $y = f(x)$ from a to b about the x-axis as such:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx \tag{36}$$

where $\sqrt{1 + [f'(x)]^2}$ corresponds to the height. Note that we are approximating the original curve following the same idea/method used to obtain the arc length.

Remark 8.1. If $x = g(y)$ is rotated about the y-axis, then we obtain...

$$S = \int_c^d 2\pi y\sqrt{1 + [f'(y)]^2}dy \tag{37}$$

11 Infinite Sequences and Series

11.1 Sequences

A **sequence** is a list of numbers written in a definite order. It can be pictured by plotting its elements on the number line. Note that we denote the sequence $\{a_1, a_2, ...\}$ as $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

- It's a function whose domain is the set of all positive integers.
- It's graph contains isolated points.

Definition 11.1. A **limit of a sequence**, L, is defined as

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty \tag{38}$$

The idea is that we can make a_n as close to L as we'd like by taking n, the index, to be sufficiently large.

- We say that a series is **convergent** if the limit of a_n exists as a finite number.
- We say that a series is **divergent** if the limit does not exist as a finite number.
- Remark: the only difference between the limit of a sequence and the limit of a function is that as n tends to infinity, n is required to be an integer. A consequence of this:

  Theorem 5. *If* $\lim_{x\to+\infty} a_n = L$ *and if* $f(n) = a_n$*, then*

  $$\lim_{n\to\infty} a_n = L$$

  This implies that we can use limit laws on sequences.

Properties of sequences: if a_n and b_n are convergent series, then we obtain the following rules:

1. $\lim_{n\to+\infty}(a_n \pm b_n) = \lim_{n\to\infty}(a_n) \pm \lim_{n\to\infty}(b_n)$

2. $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$
3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$
4. **Very useful**: $\lim_{n\to\infty}((a_n)^p) = (\lim_{n\to\infty} a_n)^p$; this property basically tells you do what you want.

Squeeze Theorem: another important theorem. If $a_n \leq b_n \leq c_n$ for some $n \geq n_0$, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L, \text{ then } \lim_{n\to\infty} b_n = L$$

Remark 11.1. If $|a_n| \to 0$ as $n \to \infty$, then $a_n \to 0$.

Definition 11.2. We say that a sequence is **increasing** if $a_n < a_{n+1} \forall n \geq 1$. We say that a function is **decreasing** if the reverse equality holds. A function that is EITHER decreasing or increasing is **monotonic**. You prove this in two ways:

- Induction (prove for a base case – the lower limit of the interval), then prove that $P(n) \to P(n+1)$.
- First derivative test.

Definition 11.3. We say that a sequence is **bounded above** if there is a number N such that $a_n \leq N, \forall n \geq 1$.

We say that a sequence is **bounded below** if there exists a number m such that $a_n > m \forall n \geq 1$. A function that is bounded BOTH above and below is called **bounded**.

Remark 11.2. Not all bounded sequences are convergent, and not all convergent sequences are bounded.

Theorem 6. *(Monotonic Sequence Theorem) Every* ***bounded*** *and* ***monotonic*** *sequence is* ***convergent****.*

11.2 Series

A **series** is the sum of all terms in a sequence. It is denoted by

$$\sum_{n=1}^{\infty} a_n$$

- We can study the behaviour of a series with **partial sums** by taking the limit as the partial sums tend to a_n.
- The partial sum is denoted as s_n.

Definition 11.4. Given a series $\sum_{n=1}^{\infty} a_n$, let s_n denote the nth partial sum

$$s_n = \sum_{i=1}^{\infty}$$

If the sequence is **convergent** and if $\lim_{n\to\infty} s_n = s$ exists as a finite number, then the SERIES $\sum a_n$ is called **convergent**, and we can write:

$$\sum_{n=1}^{\infty} a_n = s$$

Else, if the sequence is **divergent**, then its corresponding series is also called **divergent**.

Some important series to know:

- **Telescopic series**: these ones, we need to use partial fraction decomposition (sometimes) to break the summand up. We can obtain the sum of the series by cancelling out the correct terms.
- The **harmonic series**, given by
$$\sum_{n=1}^{\infty} \frac{1}{n}$$
is divergent.
- The **geometric series** is a series of the form
$$a + ar + ar^2 + ar^3 + ... + ar^{n-1} = \sum_{n=1}^{\infty} ar^{n-1}, a \neq 0$$
In other words, it is a series where each term is obtained from the preceding term by multiplying it by a common **ratio r**.
  - It converges iff $|r| < 1$. The sum of the sequence is given by:
$$\sum_{i=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \tag{39}$$

Theorem 7. *If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then as $a_n \to 0$ as $n \to +\infty$.*

Remark 11.3. The converse is COMPLETELY FALSE; if the limit of a_n tends to 0, we cannot say anything (ex - harmonic series).

- *If the limit of a_n does not exist, or exists but does not equal zero, then we can conclude that the series is divergent.*
- *This is the **test for divergence**, and should be the first tool employed to test if a series cv or dv.*

- If two series, given by a_n and b_n are convergent, then so are scalar multiples, its sum, and its difference. ALL COMPONENTS MUST BE CONVERGENT FOR THIS TO APPLY.

Remark 11.4. A finite number of terms does not impact the convergence or divergence of a series; in fact, it only depends on the behaviour of a_n as n tends to infinity. This is important.

11.3 Integral Tests and Estimates of Sums

Motivation: the aim is to develop tests to enable us to determine if a series is convergent or divergent without explicitly finding the sum, because it is generally difficult to find the sum of a series.

Definition 11.5. The integral test: suppose that f is **continuous, positive,** and **decreasing** on the interval $[1, \infty]$ and let $a_n = f(n)$. Then, the series is convergent iff its corresponding improper integral is convergent.

This proves a very important lemma ("if you do not know this, then you have no chance on the final, so learn it.")

Lemma 8. *The **p-series** test defines convergence/divergence as such:*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \textit{convergent if } p > 1 \\ \textit{divergent if } p \leq 1 \end{cases} \tag{40}$$

11.3.1 Estimating the Sum of a Series

Motivation: suppose that we are able to show that a series $\sum a_n$ is convergent. We now want to find an approximation to the sum S of the series. The **remainder** is the error made when s_n, the first n terms of the series, is used as an approximation to the total sum.

- The remainder is given by

$$R_n = s - s_n = a_{n+1} + a_{n+2} + ... \tag{41}$$

- The error when estimating using the integral test is given by:

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx \tag{42}$$

11.4 Comparison Test

Idea: compare a given series with a series that we already know is convergent or divergent.

Definition 11.6. Comparison test: suppose that $\sum a_n$ and $\sum b_n$ are series with POSITIVE terms. Then,

1. If $\sum b_n$ is convergent and $a_n \leq b_n$ $\forall n$ near infinity, then $\sum a_n$ is also convergent.
2. If $\sum b_n$ is divergent and $a_n \geq b_n$ $\forall n$ near infinity, then $\sum a_n$ is also divergent.

- In using the comparison tests, we must have a known series to work with. Generally, we have two series for this: the geometric series and p-series. A question like this will be on the final.

Definition 11.7. The **limiting comparison test** allows us to compare series without having one that is bigger or smaller than the other. Requires terms to be positive. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite and positive constant (NOT 0 or positive infinity), then there are two possibilities: (1) both cv, (2) both dv.

We can basically conclude that the series have the same behaviour as n tends to infinity. Generally take b_n to be the main terms of both the denominator and numerator.

11.5 Alternating Series

Motivation: the convergence tests that we have looked at so far only apply to series with positive terms; in this type of series and the text, we will learn to deal with series whose terms are not necessarily positive.

Definition 11.8. An **alternating series** is a series whose terms alternate between positive and negative. They are in the form of:

$$a_n = (-1)^n b_n \text{ or } a_n = (-1)^{n+1} b_n$$

Where b_n is the positive sequence – that is, the absolute value of a_n. It can also be present in trigonometric functions...

$$\frac{\cos(n\pi)}{n} \equiv \sin\left(\frac{\pi}{2} + n\pi\right) \equiv (-1)^n$$

The test for this series is as follows:
If the alternating series satisfies the following general rules, then the series is convergent. If the rules are not satisfied, then we cannot say anything and need to find another test.

1. $b_{n+1} \leq b_n \forall n$
2. $\lim_{n\to+\infty} b_n \to 0$

11.6 Absolute Convergence, Root, and Ratio Tests

11.6.1 Absolute convergence

Motivation: given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + ...$$

whose terms are the **absolute value** of the terms of the original sequence.

Definition 11.9. A series is called **absolutely convergent** if the series of the absolute values is convergent.

- A series is called **conditionally convergent** if it is convergent, but not absolutely convergent. (An example of this is the alternating harmonic series).
- If a series is **absolutely convergent**, then it is **convergent**.
- This is a good way to study the series with non-positive general terms.
- Useful thing to remember:
$$|cos(n)| \leq 1$$

11.6.2 The Ratio Test

This test is useful in determining whether a given series is absolutely convergent. This is good to use when factorials occur. Three cases.

1. If
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$$
Then, the series is absolutely convergent and therefore convergent.
2. If
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1 \text{ or} L = \infty$$
then, the series is divergent.
3. "The worst one": DO NOT DRAW ANY CONCLUSIONS FROM THIS CASE, BECAUSE YOU CANNOT:
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$$
then, the ratio test is **completely inconclusive** and you need to find another test.

11.6.3 Root test

This test is good to use when we have powers of n.

1. If
$$\lim_{n\to\infty} (|a_n|)^{1/n} = L < 1$$
then the series is **absolutely convergent**.

2. If
$$\lim_{n\to\infty} (|a_n|)^{1/n} = L > 1 \text{ or } L = \infty$$
then, the series is **absolutely divergent**.

3. The worst one again: if
$$\lim_{n\to\infty} (|a_n|)^{1/n} = L = 1$$
then the test is completely **inconclusive**

11.7 Strategy for Testing Series

Classify according to its form

- If we are asked to **compute the sum**, then we know that the series must be one that can be studied using **telescopic series**, **integral test**, or **geometric series**.
- The **limiting comparison test** is similar in form to the p-series.
- Always use the **test for divergence** at the beginning.
- P-series type series will be rendered inconclusive by the root and ratio tests.
- Use the **integral test** when the summand is easy to integrate and the three conditions are met.

The final...

- 3 h long. 11-12-2017. 14h00.
- 18-long answers similar to the mid-term. Half will be series, half will be integrals.