MATH 254: ANALYSIS I (THEOREMS, DEFINITIONS, AND RESULTS FROM THE CLASS)

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ABSTRACT. The purpose of this document is to summarise Analysis 1 (Math 254).

CONTENTS

1. INTRODUCTION

Random things we proved to get a handle on how to prove things:

- $\cap_{x\in[0,1]}[0,x] = \{0\}$.
- $2^n < n!$
- Let X and Y be sets. Consider the following family of sets:
$$\{V_i \mid i \in I, V_i \subseteq Y\}$$
then, $f^{-1}(\cup_{i\in I} V_i) = \cup_{i\in I} f^{-1}(V_i)$.
- $5^n - 1$ is divisible by 4 $\forall n \geq 1$.
- **Bernoulli's Inequality**: $\forall n \in \mathbb{N}$, $x \in \mathbb{R}$, $x \geq -1$, one has:
$$(1+x)^n \geq 1 + nx \tag{1}$$
- Every non-empty subset of the natural numbers has a smallest element.

Definition 1 (Cartesian Product)**.** Let A and B be two sets. Then, their **Cartesian Product** is defined as:
$$A \times B := \{(a,b) \mid a \in A \wedge b \in B\} \tag{2}$$

Definition 2 (Function)**.** Let D, E be sets. A **function** f from D to E is a subset of the cartesian product $D \times E$ such that $\forall x \in D$, $\exists_1\ t \in E$ such that $(x,y) \in f$. In symbols, we define:
$$f(A) := \{f(x) \mid x \in A\} \tag{3}$$

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Proposition 3 (Properties of Functions)**.** Let $f : D \to E$ be a function and let $A, B \subseteq D$. Then, consider the following:

- $f(A \cup B) = f(A) \cup f(B)$ [well behaved with respect to unions]
- $f(A \cap B) \subseteq f(A) \cap f(B)$.

Definition 4 (Pre-Image)**.** Let $f : D \to E$, $A \subseteq E$. Then, the **pre-image** is defined as:

$$f^{-1}(A) := \{x \in D \mid f(x) \in A\} \tag{4}$$

Proposition 5. Let $f : D \to E$, $A, B \subseteq E$. Then:

- $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
- $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Definition 6 (Injective)**.** Let $f : D \to E$. f is said to be **injective** if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Definition 7 (Surjective)**.** Let $f : D \to E$. f is said to be **surjective** if $\forall y \in E$, $\exists x \in D$ such that $f(x) = y$.

Definition 8 (Bijective)**.** $f : D \to E$ is called **bijective** if it is surjective and injective.

Definition 9. If $f : D \to E$ is bijective, then we can define the **inverse** function $f^{-1} : E \to D$ as follows:

$$f^{-1}(y) := x \tag{5}$$

where x is a uniquely determined point in D with $f(x) = y$.

1.1. **Countability of Finite Sets.**

Definition 10 (Cardinality)**.** Let $S = \{a_1, ..., a_n\}$. Then, the **cardinality** of S, in symbols $|S|$, is the number of elements in a set S.

Theorem 11. Let A, B be finite sets. Then, $|A| \leq |B| \iff$ there exists a function $f : A \to B$ which is injective.

Theorem 12. Let A, B be finite sets. Then, $|A| \geq |B| \iff \exists$ a surjective map from $A \to B$.

Theorem 13. Let A, B be finite sets. Then, $|A| = |B| \iff \exists$ a bijective map $f : A \to B$.

Definition 14. Let A and B be sets, not necessarily finite. We then say that A and B have the **same cardinality**, in symbols,

$$|A| = |B| \tag{6}$$

if $\exists$ a bijective map $f : A \to B$.

Theorem 15 (Cantor's Theorem)**.** Let A and B be sets. If $|A| \leq |B|$ and if $|B| \leq |A|$, then $|A| = |B|$.

Definition 16 (Countability)**.** We say that a set A with $|A| = |\mathbb{N}|$ is **countably infinite**. A set which is either finite or countably infinite is called **countable**.

Theorem 17 (Arithmetic-Geometric Inequality)**.** $\forall n \geq 1$ and for all $x_1, ..., x_n > 0$, the following holds:

$$\frac{x_1 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \tag{7}$$

Lemma 18. Let $n \in \mathbb{N}$ and let $x_1, ..., x_n > 0$. If $x_1 \cdots x_n = 1$, then:

$$x_1 + ... + x_n \geq n \tag{8}$$

Theorem 19. Let $S \subseteq \mathbb{N}$. Then, there are only two possibilities:

(1) S is finite.
(2) S is countably infinite.

Lemma 20. Let $a_1 < a_2 < \cdots$ be a strictly increasing sequence of natural numbers. Then, we can say something about the growth rate:

$$a_n \geq n \tag{9}$$

$\forall n \in \mathbb{N}$.

Theorem 21. Let $f : \mathbb{N} \to S$ be surjective. Then, S is countable.

Theorem 22 (Cantor)**.** The set $\mathbb{Q}$ of all rational numbers is countably infinite.

Theorem 23. $\mathbb{R}$ is uncountable (i.e, $\mathbb{R}$ is infinite and there does not exist a bijection from $\mathbb{N}$ to $\mathbb{R}$.

Definition 24 (Absolute Value)**.** Let $x \in \mathbb{R}$. Then, the **absolute value** of x is defined as:

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{10}$$

Note that $|x|$ is used to measure distances.

Proposition 25 (Properties of Absolute Value)**.** (1) $\forall x \in \mathbb{R}$, $|x| \geq 0$ and $|x| = 0 \iff x = 0$.
(2) $\forall x, y \in \mathbb{R}$, $|xy| = |x||y|$. Especially, $|-x| = |x|$, in this case you would simply set $y = -1$.
(3) $\forall x \in \mathbb{R}$, $-|x| \leq x \leq |x|$.
(4) Let $a > 0$, $x \in \mathbb{R}$. Then, $|x| \leq a \iff -a \leq x \leq a$.

Theorem 26 (Triangle Inequality)**.** Let $x, y \in \mathbb{R}$. Then:
(1) $|x + y| \leq |x| + |y|$
(2) $|x - y| \geq ||x| - |y||$
(3) Especially,
  (a) $|x - y| \geq |x| - |y|$
  (b) $|x - y| \geq |y| - |x|$

Corollary 27. We also have,
(1) $|x - y| \leq |x| + |y|$
(2) $|x + y| \geq |x| - |y|$ and $|x + y| \geq |y| - |x|$.

Corollary 28 (Generalisation of the Triangle Inequality)**.**

$$|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n| \tag{11}$$

Definition 29. ε-neighbourhood Let $x \in \mathbb{R}$ and let $\varepsilon > 0$ be fixed. Then, the **ε-neighbourhood** of x, $V_\varepsilon(x)$, to be:

$$\begin{aligned} V_\varepsilon(x) &:=]x - \varepsilon, x + \varepsilon[\\ &= \{y \in \mathbb{R} \mid |y - x| < \varepsilon\} \end{aligned}$$

Theorem 30. Let $x, y \in \mathbb{R}$, where $x \neq y$. Then, "x and y can be separated by neighbourhoods", i.e., $\exists$ a $\varepsilon > 0$ such that $V_\varepsilon(x) \cap V_\varepsilon(y) \neq \varnothing$.

1.2. Supremum and Infimum.

Definition 31 (Bounded From Above)**.** Let $S \subseteq \mathbb{R}$, $S \neq \varnothing$. We say that S is **bounded from above** if $\exists$ a $u \in \mathbb{R}$ such that $\forall s \in S$ $s \leq u$.

Definition 32 (Bounded from Below)**.** Let $S \subseteq \mathbb{R}$, $S \neq \varnothing$. We say that S is **bounded from below** if $\exists$ a $u \in \mathbb{R}$ such that $\forall s \in S$, $u \leq s$.

Definition 33 (Supremum/Least Upper Bound)**.** Let $S \subseteq \mathbb{R}$, $S \neq \varnothing$. $u \in \mathbb{R}$ is called a **supremum** or **least upper bound**, denoted by $\sup S$, if:
(1) u is an upper bound for S.
(2) If v is any other upper bound for S, then $u \leq v$.

If $u = \sup S \in S$, then we say that u is the **maximum element** of S.

Definition 34 (Infimum/Greatest Lower Bound)**.** Let $S \subseteq \mathbb{R}$, $S \neq \varnothing$. $u \in \mathbb{R}$ is called a **infimum** or **greatest lower bound**, denoted by $\inf S$, if:
(1) u is a lower bound.
(2) If v is an arbitrary lower bound of S, then $v \leq u$.

If $u = \inf S \in S$, then we say that u is the **minimum element of S**.

[Begin Tutorial]

Proposition 35. If $X_1, ..., X_{n+1}$ are countable sets, then so is $X_1 \times \cdots \times X_{n+1}$.

Definition 36 (Power Set)**.** Let X be a set, possibly empty. Then, the **power set of X**, denoted $\mathcal{P}(X)$, is defined as the set of all subsets of X:

$$\mathcal{P}(X) := \{A \mid A \subseteq X\} \tag{12}$$

Theorem 37 (Cantor's Theorem)**.** Let X be a set. Then, there does not exist a surjection $X \to \mathcal{P}(X)$, which means that $|X| < |\mathcal{P}(X)|$

Corollary 38 (Russel's Paradox)**.** The set of all sets does not exist.

Proposition 39. A binary sequence is a list of points

$$a_1, a_2, ..., a_n, ...$$

such that each $a_i \in \{0, 1\}$. Let $\mathcal{B}$ be the set of all binary sequences. Then, $\mathcal{B}$ is uncountable.

[End Tutorial]

Theorem 40. Let S be a non-empty and bounded set from above, with supremum $\sup S$. Define:

$$a + S := \{a + s \mid s \in S\}$$

Then, $a + S$ has a supremum which is given by:

$$\sup (a + S) = a + \sup S \tag{13}$$

Theorem 41. Let $S \neq \varnothing$, $S \subseteq \mathbb{R}$, S bounded from above with supremum $\sup S$. Let $k > 0$ and define:

$$k \cdot s := \{ks \mid s \in S\}$$

Then,

- If $k > 0$, $k \cdot S$ is bounded from above and
$$\sup k \cdot S = k \cdot \sup S \tag{14}$$
- if $k < 0$, then $k \cdot S$ is bounded from below and
$$\inf k \cdot S = k \cdot \sup S \tag{15}$$

AXIOM: we assume $\mathbb{R}$ is complete. This means that every non-empty subset $S \subseteq \mathbb{R}$ which is bounded from above has a supremum in $\mathbb{R}$.

Theorem 42 (Archimedean Property of $\mathbb{R}$)**.** Let $x \in \mathbb{R}$, $x > 0$. Then, $\exists n \in \mathbb{N}$ such that $n \geq x$.

Theorem 43. Let $x < y$, $x, y \in \mathbb{R}$. Then, $\exists r \in \mathbb{Q}$ such that $x < r < y$. I.e., this means that the rational numbers are **dense** in $\mathbb{R}$.

Theorem 44. The irrational numbers are dense in $\mathbb{R}$.

Definition 45. Let $I_1, , I_2, I_3, ...$ be intervals with the following property:

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

Then, we call the $I_1, I_2, I_3, ...$ a **nested sequence** of intervals.

Theorem 46 (Nested Interval Property)**.** Let $I_1 \supseteq I_2 \supseteq I_3 \cdots$ be a nested sequence of non-empty, closed and bounded (we call this compact) intervals, then:

$$\bigcap_{n \in \mathbb{N}} I_n \neq \varnothing \tag{16}$$

THE NESTED INTERVAL PROPERTY IS IN FACT EQUIVALENT TO COMPLETNEESS.

Corollary 47. $\mathbb{R}$ is uncountable.

[Begin Tutorial]

COMPLETENESS PROPERTY OF $\mathbb{R}$: Let X be a non-empty subset of $\mathbb{R}$ that is bounded from above. Then, X has a least upper bound, denoted by $\sup X$.

Proposition 48. Let $X \subseteq \mathbb{R}$.

(1) if X has a supremum, then X is non-empty and bounded from above.
(2) if X has an infimum, then X is non-empty and bounded from below.

Proposition 49. Let X be a non-empty set and let s be an upper bound for X in $\mathbb{R}$. Then, the following statements are equivalent:

(1) $s = \sup S$
(2) $\forall\ \varepsilon > 0$, $\exists\ x_\varepsilon \in X$ such that:

$$s - \varepsilon < x_\varepsilon \leq s \tag{17}$$

Proposition 50. Let X be a non-empty set and let v be a lower bound for X in $\mathbb{R}$. Then, the following statements are equivalent:

(1) $v = \inf S$
(2) $\forall\ \varepsilon > 0$, $\exists\ x_\varepsilon \in X$ such that:

$$v \leq x_\varepsilon < v + \varepsilon \tag{18}$$

A useful application of the Archimedean property: $\forall\ \varepsilon > 0$, one has that $\exists$ an $m \in \mathbb{N}$ such that $0 < \frac{1}{m} < \varepsilon$.

Theorem 51 (Characterisation of Intervals)**.** Let $S \subseteq \mathbb{R}$ contain at least two points and assume that S satisfies the property:

$$x, y \in S \text{ and } x < y \ \Rightarrow [x, y] \subseteq S \tag{19}$$

then S is an interval.

Proposition 52 (Algebraic Properties of Sup and Inf)**.** Let A, B be non-empty subsets of $\mathbb{R}$ that are bounded from above. Suppose that both $x, y \in [0, \infty[$. Then:

(1) $\sup(A \cdot B) = \sup(A)\sup(B)$, where $A \cdot B := \{ab \mid a \in A, b \in B\}$.

[End Tutorial]

2. Point-Set Topology

Definition 53 (Open)**.** A set $U \subseteq \mathbb{R}$ is called **open** if $\forall x \in U$, $\exists \varepsilon > 0$ such that $V_\varepsilon(x) \subseteq U$.

Definition 54 (Closed)**.** A set $A \subseteq \mathbb{R}$ is called **closed** if its complement, $\mathbb{R} \setminus A$, is open.

Theorem 55. $\forall x \in \mathbb{R}$, $\forall \varepsilon > 0$, $V_\varepsilon(x)$ is open.

Theorem 56. Open intervals are open "seems self-evident, but still requires proof."

Theorem 57. All closed intervals are closed.

Theorem 58. Let J be an arbitrary index set and let U_j be open, $U_j \subseteq \mathbb{R}$, $\forall j \in J$. Then, the union is open:

$$U := \bigcup_{j \in J} U_j \tag{20}$$

Remark 59. Arbitrary intersections of open sets are, in general, not open.

Theorem 60. The finite intersection of open sets are open, i.e., if $U_1, ..., U_n \subseteq \mathbb{R}$ are open, then:

$$U := \bigcap_{i=1}^{n} U_i = U_1 \cap U_2 \cap \cdots U_n \tag{21}$$

is open.

Theorem 61. The arbitrary intersection of closed sets are closed, i.e., if J is some index set, and if A_j is closed for each $j \in J$, then:

$$A := \bigcap_{j \in J} A_j \tag{22}$$

is closed.

Theorem 62. Finite unions of closed sets are closed.

Theorem 63. $\varnothing$ and $\mathbb{R}$ are the only subsets of $\mathbb{R}$ that are both open and closed.

Definition 64 (Boundary Point)**.** Let $U \subseteq \mathbb{R}$, $x \in \mathbb{R}$ is called a **boundary point of U** if, $\forall \varepsilon > 0$, $V_\varepsilon(x) \cap U \neq \varnothing$ and $V_\varepsilon(x) \cap (\mathbb{R} \setminus U) \neq \varnothing$

Definition 65. The set of all boundary points of a subset $U \subseteq \mathbb{R}$ is called the **boundary** of U, denoted ∂U.

Theorem 66. Let $S \subseteq \mathbb{R}$ and $U \subseteq S$, U open. Then, $U \cap \partial S = \varnothing$.

Theorem 67. Let $S \subseteq \mathbb{R}$. Then, $\partial S = \partial(\mathbb{R} \setminus S)$.

Theorem 68. Let $S \subseteq \mathbb{R}$. Then, ∂S is closed.

Theorem 69. Let $S \subseteq \mathbb{R}$. Then,

(1) S is open $\iff$ S contains *none* of its boundary points, i.e.,

$$S \cap \partial S = \varnothing \qquad \text{or} \qquad \partial S \subseteq \mathbb{R} \setminus S \tag{23}$$

(2) S is closed $\iff$ S contains all of its boundary points, i.e.:

$$\partial S \subseteq S \tag{24}$$

Definition 70 (Interior)**.** Let $S \subseteq \mathbb{R}$. Then, the **interior** $\text{int}(S)$ is defined as:

$$\text{int}(S) := \bigcup_{U \subseteq S, U \text{open}} U \tag{25}$$

By definition, the interior is the largest open set contained in S.

Definition 71 (Closure)**.** Let $S \subseteq \mathbb{R}$. The **closure**, denote $\overline{S} := \text{cl}(S)$ is:

$$\overline{S} := \bigcap_{A \supseteq S} A \tag{26}$$

which is closed since arbitrary intersections of closed sets are closed. By definition, the closure is the smallest closed set containing S.

Proposition 72. (1) S open $\iff$ $\text{int}(S) = S$.

(2) S closed $\iff$ $\overline{S} = S$.

(3) $S \subseteq T \Rightarrow \overline{S} \subseteq \overline{T}$ and $\text{int}(S) \subseteq \text{int}(T)$.

[Begin Tutorial]

Theorem 73 (Characterisation of Intervals)**.** Let $I \subseteq \mathbb{R}$ containing at least two points. Assume that I satisfies the following property: if $x, y \in I$ with $x < y$, then $[x, y] \subseteq I$. Then, we say that I is an interval.

[End Tutorial]

Proposition 74. Properties:

(1) If $S \subseteq T$, S open, then $S \subseteq \text{int}(T)$.
(2) If $S \subseteq T$, T closed, then $\overline{S} \subseteq T$.
(3) $\overline{\overline{S}} = \overline{S}$.
(4) $\text{int}(\text{int}(S)) = \text{int}(S)$.
    (a) CAUTION! In general, $\partial(\partial S) \neq \partial S$ in general.
(5) $\text{int}(S) \cup \partial S = \overline{S}$.

Theorem 75 (Characterisation of Open intervals in $\mathbb{R}$)**.** A subset $S \subseteq \mathbb{R}$ is open $\iff$ S is the countable union of open intervals.

3. Sequences

Definition 76. An **infinite sequence** is a function $f : \mathbb{N} \to \mathbb{R}$ for which $n \mapsto f(n) = a_n$.

Definition 77. Let (a_n) be a sequence, $L \in \mathbb{R}$. We say that (a_n) **converges** to L, or that the **limit** of (a_n) is L, if:

$$\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N},\ \text{s.t. } \forall\ n \geq N,\ |a_n - L| < \varepsilon \tag{27}$$

Theorem 78. Let (a_n) be a sequence. If (a_n) converges, then the limit is uniquely determined.

3.1. Some Results on Convergent Sequences.

Theorem 79. Every convergent sequence is bounded.

Theorem 80. Let (a_n), (b_n) be convergent sequences with $a := \lim(a_n)$ and $b := \lim(b_n)$. Then,

1. $(a_n + b_n)$ is convergent and $\lim(a_n + b_n) = a + b$.
2. $(a_n \cdot b_n)$ is convergent and $\lim(a_n \cdot b_n) = a \cdot b$.

Corollary 81.
1. Let $c \in \mathbb{R}$, (a_n) convergent with $a = \lim(a_n)$. Then, $c(a_n)$ is convergent with $\lim(c \cdot a_n) = ca$.
2. (a_n), (b_n) convergent with $a = \lim(a_n)$, $b = \lim(b_n)$. Then, $(a_n - b_n)$ is convergent and $\lim(a_n - b_n) = a - b$.

Theorem 82. Let (b_n) be convergent, $b := \lim(b_n)$ such that $\forall n \in \mathbb{N}$, $b_n \neq 0$ and $b \neq 0$. Then, $(1/b_n)$ converges and its limit is $1/b$.

Theorem 83. Let (a_n), (b_n) be convergent sequences with $a := \lim(a_n)$, $b := \lim(b_n)$ and $\forall n \in \mathbb{N}$, $b_n \neq 0$. Then, (a_n/b_n) converges and $\lim(a_n/b_n) = (a/b)$.

Theorem 84 (Convergence Criterion)**.** Let (a_n) be a sequence, (b_n) a convergent non-negative sequence with $\lim(b_n) = 0$, and let $c > 0$. If $\exists k \in \mathbb{N}$ such that $\forall n \geq k$, $|a_n - a| \leq cb_n$, then (a_n) converges and $\lim(a_n) = a$.

Theorem 85. Let (x_n) be a sequence such that $\exists k \in \mathbb{N}$, $\forall n \geq k$, $x_n \geq 0$. If (x_n) converges, then $x := \lim(x_n) \geq 0$.

Corollary 86. Let (x_n), (y_n) be convergent sequences with $k \in \mathbb{N}$ such that $x_n \leq y_n$ $\forall n \geq k$. Then, $\lim(x_n) \leq \lim(y_n)$.

Corollary 87. Let (x_n) be a convergent sequence such that $\exists\ k \in \mathbb{N}$ such that $\forall n \geq k$, $a \leq x_n \leq b$, $a, b \in \mathbb{R}$. Then, $a \leq \lim(x_n) \leq b$.

Theorem 88 (Squeeze Theorem)**.** Let (a_n), (b_n), (x_n) be sequences with $\exists k \in \mathbb{N}$ such that $\forall n \geq k$, we have $a_n \leq x_n \leq b_N$. Furthermore, let (a_n) and (b_n) converge to the same limit x. Then,

1. (x_n) converges and
2. $\lim(x_n) = x$.

Theorem 89. Assume that (a_n) is bounded and that (b_n) converges to zero. Then, $(a_n \cdot b_n)$ converges to zero.

3.2. Monotone Sequences.

Definition 90 (Increasing, strictly increasing, eventually increasing)**.** Let (x_n) be a sequence. Then,

1. (x_n) is **increasing** if $x_1 \leq x_2 \leq \ldots$
2. (x_n) is **strictly increasing** if $x_1 < x_2 < \ldots$
3. (x_n) is **eventually increasing** if $\exists\ k \in \mathbb{N}$ such that $x_k \leq x_{k+1} \leq x_{k+2} \leq \ldots$

Definition 91 (Monotone)**.** A sequence (x_n) is called **monotone** if it is increasing or decreasing.

Theorem 92 (Monotone Sequence Theorem)**.** Let (x_n) be a monotone sequence.

1. (x_n) converges $\iff$ it is bounded.

(2) If (x_n) is bounded and increasing, then

$$\lim(x_n) = \sup\{x_n \mid n \in \mathbb{N}\} \tag{28}$$

(3) if (x_n) is bounded and decreasing, then

$$\lim(x_n) = \inf\{x_n \mid n \in \mathbb{N}\} \tag{29}$$

[Begin Tutorial]

Proposition 93. Let $(x_n) \to x \in \mathbb{R}$ be a sequence. Then, $(|x_n|) \to |x|$.

Theorem 94. Let $a > 1$. Then, $\lim(1/a^n) = 0$.

Theorem 95. Let $a \in]-1,1[$. Then, $\lim(a^n) = 0$.

Theorem 96. Let (x_n) be with $x_n > 0$. If

$$L = \lim\left(\frac{x_{n+1}}{x_n}\right) \tag{30}$$

exists and $L < 1$, then $\lim(x_n) = 0$.

Definition 97 (Series)**.** Let (x_n) be a sequence in $\mathbb{R}$ or $\mathbb{C}$. For $N \in \mathbb{N}$, define:

$$S_N := \sum_{n=1}^{N} x_n \tag{31}$$

Thus, (S_n) is a sequence in $\mathbb{R}$ or $\mathbb{C}$. If $\lim_{N\to\infty} S_N =: S$ exists, we write $\sum_{n=1}^{\infty} x_n$.

Definition 98 (Converge, Series)**.** We say that $\sum_{n=1}^{\infty} |x_n| = \lim_{N\to\infty} \sum_{n=1}^{N} |x_n|$ exists $\iff$ the sequence of partial sums is bounded.

Example 99. $\lim(2^n/n!) = 0$.

Example 100. $\lim(n!/n^n) = 0$.

[End Tutorial]

3.3. **Subsequences.**

Definition 101. Let $n_1 < n_2 < n_3 < ...$ be natural numbers. Let (x_n) be a sequence and consider:

$$(x_{n_k}) = (x_{n_1}, x_{n_2}, ...) \tag{32}$$

The (x_{n_k}) is a **subsequence** of (x_n).

Theorem 102. Let $(x_n) \to x$ and let (x_{n_k}) be a subsequence. Then, (x_{n_k}) converges to x.

Corollary 103. Let (x_n) be a sequence. Then, (x_n) converges $\iff$ all subsequences of (x_n) converge to the *same* limit.

Example 104. $\lim(1 + a/n)^n = e^a$.

Example 105. $\lim(\sqrt[n]{a}) = 1$ for $a > 1$, $n \in \mathbb{N}$.

Example 106. $\lim(\sqrt[n]{n}) = 1$.

Definition 107 (Accumulation Point)**.** Let (x_n) be a sequence. A point $x \in \mathbb{R}$ is called an **accumulation point** of x_n if $\exists$ a subsequence (x_{n_k}) of x_n that converges to x.

Theorem 108. Let (x_n) be a sequence, $x \in \mathbb{R}$ an accumulation point of (x_n) $\iff$ $\forall \varepsilon > 0$, $V_\varepsilon(x)$ contains infinitely many points of (x_n).

Theorem 109 (Bolzano-Weierstrass Theorem)**.** Let (x_n) be a bounded sequence in $\mathbb{R}$. Then, (x_n) has a convergent subsequence i.e., (x_n) has at least one accumulation point.

Definition 110 (Limit Superior)**.** Let (x_n) be bounded. The greatest accumulation point of (x_n) is called the **limit superior** of (x_n): $x^* := \limsup(x_n)$.

Definition 111 (Limit inferior)**.** Let (x_n) be bounded. The smallest accumulation point of (x_n) is called the **limit inferior** of (x_n): $x_* := \liminf(x_n)$.

Theorem 112. Let (x_n) be bounded. Let $v_m := \sup(x_1, ..., x_m)$. Then,

$$\begin{aligned}\lim(v_m) &= \lim(\sup\{x_n \mid n \geq m\}) \\ &= \limsup(x_n)\end{aligned}$$

and

$$\liminf(x_n) = \lim(\inf\{x_n \mid n \geq m\})$$

3.4. **Cauchy Sequences.**

Definition 113 (Cauchy Sequence)**.** A sequence (x_n) is called a **Cauchy sequence** if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$, one has

$$|x_n - x_m| < \varepsilon \tag{33}$$

Theorem 114. A sequence in $\mathbb{R}$ converges $\iff$ it is a Cauchy Sequence.

Theorem 115. Every Cauchy Sequence is bounded.

Definition 116 (Contractive Sequence)**.** A sequence (x_n) is **contractive** if $\exists$ a $0 < c < 1$ such that $\forall n \in \mathbb{N}$,

$$|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n| \tag{34}$$

Theorem 117. Every contractive sequence is Cauchy, and thus converges.

3.5. **Divergence to $\pm\infty$.**

Definition 118. Let (x_n) be a sequence.

(1) (x_n) **diverges to ∞** if $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $x_n > M$.
(2) (x_n) **diverges to $-\infty$** if $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $x_n < M$.

Theorem 119. An increasing sequence diverges to $+\infty$ $\iff$ it is unbounded. Similarly, a decreasing sequence diverges to $-\infty$ $\iff$ it is unbounded.

[Begin Tutorial]

Theorem 120. Let $F \subseteq \mathbb{R}$, $F \neq \varnothing$. Then, TFAE:

(1) F is closed.
(2) If x_n is a sequence in F and $x = \lim(x_n)$, then $x \in F$.

Proposition 121. Let (x_n) be a bounded sequence. Then, $\lim(x_n)$ exists $\iff$ (x_n) has only one accumulation point.

Proposition 122. Let (x_n) be bounded, Then, $\lim(x_n)$ exists $\iff$ $\limsup(x_n) = \liminf(x_n)$.

[End Tutorial]

4. Limits of Functions

Definition 123. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let $c, L \in \mathbb{R}$. We say that the **limit of f as x approaches c is L**, in symbols, $\lim_{x\to x} f(x) = L$, if $\forall$ sequences $(x_n) \in A$ with $\lim(x_n) = c$, $\lim(f(x_n)) = L$.

Definition 124 (Cluster Point)**.** Let $A \subseteq \mathbb{R}$. c is called a **cluster point** of A if either of the two equivalent definitions hold:

(1) There exists a sequence $(x_n) \in A \setminus \{c\}$ such that $\lim(x_n) = c$.
(2) $\forall \varepsilon > 0$, $V_\varepsilon^*(c) \cap A \neq \varnothing$.

Theorem 125. Let $A \subseteq \mathbb{R}$, c a cluster point of A. Let $f : A \to \mathbb{R}$. If $\lim_{x\to c}(f(x))$ exists, then it is uniquely determined.

Definition 126. A point $c \in A$ which is not a cluster point is called an **isolated point**, i.e., c is isolated if $\exists \varepsilon > 0$ such that $V_\varepsilon^*(c) \cap A \neq \varnothing$.

Theorem 127. Let $A \subseteq \mathbb{R}$, c a cluster point of A. Then, $c \in \overline{A} = A \cup \partial A$.

Definition 128 ($\varepsilon - \delta$ definition of a limit)**.** Let $f : A \to \mathbb{R}$, c a cluster point of A, $L \in \mathbb{R}$. We say that $\lim_{x \to c} f(x) = L$ if:

$$\forall \varepsilon > 0, \exists \delta > 0 \ s.t. \ \forall x \in A, \ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \tag{35}$$

Definition 129 (Topological Definition of a Limit)**.** Two equivalent definitions:

(1) $\forall \varepsilon > 0$, $\exists \ \delta > 0$ such that $\forall x \in V_\delta^*(c)$, $f(x) \in V_\varepsilon(L)$.
(2) $\forall \varepsilon > 0$, $\exists \delta > 0$. such that $f(V_\delta^*(c)) \subseteq V_\varepsilon(L)$.

Theorem 130. The sequential definition and the $\varepsilon - \delta$ definition of a limit are equivalent.

Theorem 131 (Sequential Criterion for the non-existence of a limit)**.** $f : A \to \mathbb{R}$, c a cluster point of A. Then,

(1) Let (x_n) be a sequence in $A \setminus \{c\}$ with $\lim(x_n) = c$. If $(f(x_n))$ diverges, then $\lim_{x \to c} f(x)$ does not exist.
(2) Let (x_n), (y_n) be sequences in $A \setminus \{c\}$ with $\lim(x_n) = c = \lim(y_n)$. If $(f(x_n))$ and $(f(y_n))$ both converge but have different limits, then $\lim_{x \to c} f(x)$ does not exist.

Theorem 132 (Limit Laws)**.** Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$, c a cluster point of A such that $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exists. Then:

(1) $\lim_{x \to c}[af(x) + bg(x)] = a \lim_{x \to c} f(x) + b \lim_{x \to c} g(x)$.
(2) $\lim_{x \to c}[f(x)g(x)] = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$
(3) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$

4.1. **Continuity.**

Definition 133 (Continuous)**.** Let $f : A \to \mathbb{R}$, c a cluster point of A, $c \in A$. We say that f is **continuous at c** if:

$$\lim_{x \to c} f(x) = f(c) \tag{36}$$

Theorem 134. Let $f : A \to \mathbb{R}$, c a cluster point of A, $a, b \in \mathbb{R}$ such that $a \leq f(x) \leq b \ \forall x \in A$. If $\lim_{x \to c} f(x)$ exist, then it holds that

$$a \leq \lim_{x \to c} f(x) \leq b \tag{37}$$

Theorem 135 (Squeeze Theorem)**.** Let f, g, h be functions from $A \to \mathbb{R}$, let c be a cluster point of A such that $\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x)$ and $g(x) \leq f(x) \leq h(x) \ \forall x \in A$. Then, $\lim_{x \to c} f(x)$ exists and equals L.

Theorem 136. Let $f : A \to \mathbb{R}$, c a cluster point of A. Then, $\lim_{x \to c} f(x)$ exists $\iff$ both one-sided limits exist and are equal.

Definition 137 (Sequential Definition of Continuity)**.** $\lim_{x \to c} f(x) = f(c)$ if $\forall (x_n)$ in $A \setminus \{c\}$ with $\lim(x_n) = c$, it follows that $\lim(f(x_n)) = f(c)$.

Theorem 138. Let $f, g : A \to \mathbb{R}$ be continuous, c a cluster point, $c \in A$, f, g continuous at c. Then:

(1) $f + g$ is continuous at c.
(2) $f - g$ is continuous at c
(3) $f \cdot g$ is continuous at c.
(4) f/g is continuous at c, provided $g(x) \neq 0 \ \forall x \in A$.

Theorem 139. Let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, $f(A) \subseteq B$, f continuous at $c \in A$, g continuous at $d := f(c)$. Then, $g \circ f : A \to \mathbb{R}$ is continuous at c.

Theorem 140 (Location of Roots Theorem)**.** Let $I := [a,b]$, $f : I \to \mathbb{R}$ be continuous such that $f(a) > 0$ and $f(b) < 0$ or vice versa. Then, $\exists\ c \in]a,b[$ such that $f(c) = 0$.

Theorem 141 (Intermediate Value Theorem)**.** Let $f : I \to \mathbb{R}$, f continuous on I. Let $a,b \in I$ with $f(a) < f(b)$ and let d be a point in between. Then, $\exists$ a c between a and b with $f(c) = d$.

Theorem 142 (Preservation of Intervals)**.** Let $f : A \to \mathbb{R}$ continuous, $I \subseteq A$. Then, $f(I)$ is an interval.

Definition 143 (Open Cover)**.** Let $S \subseteq \mathbb{R}$, $\mathcal{C} := \{U_i \mid i \in I\}$ a collection of open sets such that $S \subseteq \bigcup_{i\in I} U_i$. Then, we say that $\mathcal{C}$ is an **open cover** for S.

Theorem 144 (Heine-Borel)**.** A subset $S \subseteq \mathbb{R}$ is compact $\iff$ every open cover of S has a finite sub-cover.

Theorem 145. Let $A \subseteq \mathbb{R}$ be compact, $f : A \to \mathbb{R}$ locally bounded on A. Then, f is bounded on A.

Theorem 146 (Topological Characterisation of Continuity)**.** Let $f : A \to \mathbb{R}$; f is continuous on A $\iff$ the pre-image under f of every open set is open in A.

Definition 147 (Relatively Open)**.** $W \subseteq \mathbb{R}$ is called **open in A** if there exists an open set $U \subseteq \mathbb{R}$ such that $W = A \cap U$.

Theorem 148. Let $f : A \to \mathbb{R}$ be continuous and let A be compact. Then, $f(A)$ is compact.

Corollary 149. Let $f : [a,b] \to \mathbb{R}$ be continuous. Then, $f([a,b])$ is a compact interval.

Theorem 150 (Min-Max Theorem)**.** Let $f : A \to \mathbb{R}$ be continuous; A compact. Then, f has at least one minimum and one maximum.

Definition 151 (Uniformly Continuous)**.** A function $f : A \to \mathbb{R}$ is **uniformly continuous** on A if $\forall \varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon) > 0$ such that $\forall u, x \in A$ such that $|x - u| < \delta$ implies $|f(x) - f(u)| < \varepsilon$.

Theorem 152 (Two-Sequence Criterion for Non-Uniform Continuity)**.** Let $f : A \to \mathbb{R}$. If $\exists\ \varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ but $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$, then f is not uniformly continuous.

Theorem 153. Let $f : A \to \mathbb{R}$ be uniformly continuous. Let (x_n) be a Cauchy sequence in A. Then, $(f(x_n))$ is also a Cauchy sequence.

Theorem 154. Let $f : A \to \mathbb{R}$, f continuous, A compact. Then, f is uniformly continuous on A.

Definition 155. $f : A \to \mathbb{R}$ is called a **Lipschitz Function** or is said to be **Lipschitz Continuous** or is said to satisfy a **Lipschitz Condition** if $\exists k > 0$ such that $|f(x) - f(u)| \leq k|x - u|$ for all $u, x \in A$.

Theorem 156. Let $f : A \to \mathbb{R}$. If f is **Lipschitz continuous** on A, then f is uniformly continuous on A.

5. Differentiation

Definition 157. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$. Let $c \in I$. We say that f is **differentiable** at a if the following limit exists:

$$f'(c) := \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \tag{38}$$

Theorem 158 (Caratheodory)**.** Let $f : I \to \mathbb{R}$, $c \in I$, f is differentiable at c $\iff$ there exists a $\varphi : I \to \mathbb{R}$ which is continuous at c such that $f(x) = f(c) + f(x)(x - c)$. In that case, $\varphi(c) = f'(c)$.

Theorem 159 (Chain Rule)**.** Let $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$. $f(I) \subseteq J$, $c \in I$, $d := f(c)$. Assume f is differentiable at c, g differentiable at d. Then, $g \circ f : I \to \mathbb{R}$ is differentiable at c and:

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c) \tag{39}$$

Theorem 160 (Fermat's Theorem)**.** Let $f : I \to \mathbb{R}$, $c \in I$, $c \notin \partial I$, f differentiable at c. Let f have a local extremum at c. Then, $f'(c) = 0$.

Theorem 161 (Rolle's Theorem)**.** Let $f : [a, b] \to \mathbb{R}$, f continuous on $[a, b]$ and differentiable on the open interval $]a, b[$. Let $f(c) = f(b) = 0$. Then, $\exists$ a $c \in]a, b[$ such that $f'(c) = 0$.

Theorem 162 (Mean Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$, f continuous on $[a, b]$, f differentiable on $]a, b[$. Then, $\exists$ a $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{40}$$

5.1. **Applications of the Mean Value Theorem.**

Theorem 163. Let $f : [a, b] \to \mathbb{R}$ be differentiable. Then, $f' \equiv 0$ on $[a, b] \iff f$ is constant on $[a, b]$.

Corollary 164. Let $f, g : [a, b] \to \mathbb{R}$ differentiable such that $f' \equiv g'$ on $[a, b]$. Then, $\exists$ a $c \in \mathbb{R}$ such that $g = f + c$.

Theorem 165. Let $f : [a, b] \to \mathbb{R}$, f differentiable. Then, f is increasing on $[a, b] \iff f'(x) \geq 0$ $\forall x \in [a, b]$.

Theorem 166. Let $f : I \to \mathbb{R}$ be differentiable. Then, f is Lipschitz continuous on $I \iff f'$ is bounded on I.