1 Graphs

Definition 1.1. A graph G is a pair of sets $(V(G), E(G))$ where

- $V(G)$ is the set of vertices
- $E(G)$ is the set of edges such that each edge has one or two vertices as ends. More formally, $E(G)$ is equipped with a function $\phi_G : E(G) \to \{\{u,v\} : u,v \in V(G), u \neq v\} \cup \{\{u\} : u \in V(G)\}$. For $e \in E(G)$, $\phi_G(e)$ is then called the ends of e

Definition 1.2. A loop is an edge with one end.

Definition 1.3. A pair of (distinct) edges with the same ends are parallel.

Definition 1.4. A graph is simple if it has no loops or parallel edges.

Definition 1.5. An edge joins its ends.

Definition 1.6. An edge e is incident to a vertex v if v is an end of e.

Definition 1.7. The degree of $v \in V(G)$ is the number of edges of G incident to v. Notation: $\deg(v)$ or $\deg_G(v)$.

Theorem 1.1 (Handshaking Lemma)**.** *For any graph G, $\sum_{v \in V(G)} \deg_G(v) = 2|E(G)|$.*

Definition 1.8. Two (distinct) vertices are adjacent (neighbors) if they are joined by an edge.

Definition 1.9. The null graph is the smallest graph (i.e. $G = (V(G), E(G))$ where $V(G) = \varnothing = E(G)$ is the null graph)

Definition 1.10. The simple graph on n vertices with every pair of vertices adjacent is called complete and is denoted by K_n.

Definition 1.11. A graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $V \subseteq G$.

Definition 1.12. Let $H_1, H_2 \subseteq G$. Then $H_1 \cup H_2$ where $V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$ and $E(H_1 \cup H_2) = E(H_1) \cup E(H_2)$ is a subgraph of G. We call $H_1 \cup H_2$ the union of H_1 and H_2.

Definition 1.13. Let $H_1, H_2 \subseteq G$. Then $H_1 \cap H_2$ where $V(H_1 \cap H_2) = V(H_1) \cap V(H_2)$ and $E(H_1 \cap H_2) = E(H_1) \cap E(H_2)$ is a subgraph of G. We call $H_1 \cap H_2$ the intersection of H_1 and H_2.

Definition 1.14. Two graphs G and H are isomorphic if they are the same up to relabelling vertices or, formally, if $\exists\psi : V(G) \cup E(G) \to V(H) \cup E(H)$, $\psi(V(G)) = V(H)$, $\psi(E(G)) = E(H)$, ψ bijective, $\phi_H(\psi(e)) = \phi_G(e) \forall e \in E(G)$.

Definition 1.15. A path on n vertices, denoted P_n, is a graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{e_1, \ldots, e_{n-1}\}$ such that e_i joins v_i and v_{i+1} for every $1 \leq i \leq n-1$. We say the path has ends v_1 and v_n. We say $H \subseteq G$ is a path in G if H and some path are isomorphic

Definition 1.16. A cycle on n vertices, C_n, has vertex set $\{v_1, \ldots, v_n\}$, edge set $\{e_1, \ldots, e_n\}$ such that e_i has ends v_i and v_{i+1} for every $1 \leq i \leq n-1$ and e_n has ends v_1 and v_n. We say $H \subseteq G$ is a cycle in G if H and some cycle are isomorphic.

Definition 1.17. The length of a path or cycle is the number of edges in it.

Definition 1.18. A walk in a graph G is a non-empty alternating sequence of vertices and edges of G, $v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$, such that e_i has ends v_{i-1} and v_i for $1 \leq i \leq k$. We say the walk has ends v_0 and v_k.

Definition 1.19. A walk $v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$ has length k, the number of edges in the sequence.

Definition 1.20. A walk is closed if its ends are the same.

2 Connectivity

Definition 2.1. For $u, v \in V(G), u \neq v$, u is connected to v if there exists a walk in G with ends u and v.

Lemma 2.1. *If there exists a walk in G with ends $u, v \in V(G)$ then there exists a path with ends u and v.*

Definition 2.2. A graph G is connected if any $u, v \in V(G), u \neq v$ are connected.

Lemma 2.2. *A graph G is not connected if and only if there exists a partition (X, Y), $X, Y \neq \varnothing$ of $V(G)$ such that no edge of G has one end in X and the other in Y.*

Lemma 2.3. *If $H_1, H_2 \subseteq G$ are connected and $V(H_1) \cap V(H_2) \neq \varnothing$ then $H_1 \cup H_2$ is connected.*

Definition 2.3. A (connected) component of a graph G is a maximal connected subgraph of G.

Lemma 2.4. *Every $v \in V(G)$ belongs to a unique component.*

Lemma 2.5. *$H \subseteq G$ is a component of G if and only if H is connected and $E(H)$ contains all $e \in E(G)$ with at least one end in H.*

Definition 2.4. For a graph G, comp(G) is the number of components of G (well-defined by Lemma 2.4).

Definition 2.5. For a graph G, $e \in E(G)$, $G \setminus e$ where $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$ is a graph.

Definition 2.6. For a graph G, $v \in V(G)$, $G \setminus v$ where $V(G \setminus e) = V(G) \setminus \{v\}$ and $E(G \setminus e) = E(G) \setminus \{e \in E(G) : v \in \phi_G(e)\}$ is a graph.

Definition 2.7. For a graph G, $H \subseteq G$, $G \setminus H$ where $V(G \setminus H) = V(G) \setminus V(H)$ and $E(G \setminus H) = E(G) \setminus (E(H) \cup \{e \in E(G) : \phi_G(e) \cap V(H) \neq \varnothing\})$ is a graph.

Definition 2.8. Let G be connected. $e \in E(G)$ is a cut edge of G if e is not an edge of any cycle in G.

Lemma 2.6. *Let $e \in E(G)$ have ends $u, v \in V(G)$. Then exactly one of the following holds:*

- *e is a cut edge, u, v belong to different components of $G \setminus e$, comp$(G \setminus e) =$ comp$(G) + 1$;*
- *e is not a cut edge, u, v belong to the same component of $G \setminus e$, comp$(G) =$ comp$(G \setminus e)$.*

3 Trees and Forests

Definition 3.1. A forest is a graph with no cycles ($\Leftrightarrow$ every edge in a forest is a cut edge).

Definition 3.2. A tree is a non-null connected forest.

Lemma 3.1. *Let F be a non-null forest. Then comp$(F) = |V(F)| - |E(F)|$.*

Definition 3.3. A leaf in a graph is a vertex of degree one.

Lemma 3.2. *Let T be a tree with $|V(T)| \geq 2$. Let X be the set of leaves of T and let Y be the set of vertices of degree ≥ 3. Then $|X| \geq |Y| + 2$.*

Remark. *T has at least 2 leaves.*

Lemma 3.3. *If a tree has exactly 2 leaves u, v, it is a path with ends u, v.*

Lemma 3.4. *Let v be a leaf in a tree T. Then $T \setminus v$ is a tree.*

Lemma 3.5. *Let v be a leaf in a graph G. If $G \setminus v$ is a tree then G is a tree.*

Lemma 3.6. *Let T be a tree, $u, v \in V(T)$ then there exists unique path in T with ends u, v.*

4 Spanning Trees

Definition 4.1. Let G be a graph. A tree T is a spanning tree of G if $T \subseteq G$, $V(T) = V(G)$.

Lemma 4.1. *Let G be a connected non-null graph. Let $H \subseteq G$ chosen minimal such that $V(H) = V(G)$, H connected. Then H is a spanning tree of G.*

Lemma 4.2. *Let G be a connected non-null graph. Let $H \subseteq G$ chosen maximal such that H has no cycles. Then H is a spanning tree of G.*

Definition 4.2. Let T be a spanning tree in G, let $f \in E(G) \setminus E(T)$. A fundamental cycle of f with respect to T is a cycle $C \subseteq G$ such that $f \in E(C)$ and $C \setminus f \subseteq T$ ($C \setminus f$ is a path in T).

Lemma 4.3. *Let T be a spanning tree in G. Let $f \in E(G) \setminus E(T)$. Then there exists a unique fundamental cycle of f with respect to T.*

Lemma 4.4. *Let T be a spanning tree in G. Let $f \in E(G) \setminus E(T)$, let C be the fundamental cycle of f with respect to T. Let T' be the graph obtained from T be adding f and deleting some $e \in E(C)$. Then T' is a spanning tree of G.*

Definition 4.3. Let G be a graph, let $w : E(G) \to \mathbb{R}_+$. Given a subgraph H of G define $w(H) = \sum_{e \in E(H)} w(e)$. A spanning tree T of G is the minimal spanning tree of (G, w) ($\mathrm{MST}(G, w)$) if $w(T)$ is minimal among all spanning trees of G.

Corollary 4.5. *Let G, T, f be as in Lemma 4.4. Let $w : E(G) \to \mathbb{R}_+$. If T is $MST(G, w)$ then $w(f) \geq w(e)$.*

Theorem 4.6. *Let G be a graph. Let $w : E(G) \to \mathbb{R}_+$, such that $w(e) \neq w(f)$ for any $e, f \in E(G)$. Let T be an $MST(G, w)$ and let $E(T) = \{e_1, \ldots, e_k\}$ be such that $w(e_1) < \cdots < w(e_k)$. Then for every $1 \leq i \leq k$, e_i is the edge of G with minimal weight among all edge f where $f \notin \{e_1, \ldots, e_{i-1}\}$ and where $\{e_1, \ldots, e_{i-1}, f\}$ does not contain the edge set of a cycle.*

Theorem 4.7. *Consider Kruskal's algorithm, where*

- *Input: G connected non-null graph; $w : E(G) \to \mathbb{R}_+$;*
- *For $i = 1, \dots, |V(G)| - 1$ let $e_i \in E(G)$ be chosen with $w(e_i)$ minimal among $\{f : f \notin \{e_1, \dots, e_{i-1}\}, \{e_1, \dots, e_{i-1}, f\}$ does not contain the edge set of a cycle$\}$;*
- *Output: A tree T with $V(T) = V(G)$, $E(T) = \{e_1, \dots, e_{|V(G)|-1}\}$.*

This algorithm outputs $MST(G, w)$.

Theorem 4.8 (Cayley's formula)**.** *The complete graph on n vertices has n^{n-2} spanning trees.*

5 Euler Tours and Hamiltonian Cycles

Lemma 5.1. *Let G be a graph, $E(G) \neq \varnothing$ and G has no leaves. Then G contains a cycle.*

Lemma 5.2. *Let G be a graph such that every vertex of G has even degree. Then there exists cycles $C_1, \dots, C_k$ such that $(E(C_1), \dots E(C_k))$ is a partition of $E(G)$, i.e. every edge of G belongs to exactly one of $C_i, 1 \leq i \leq k$.*

Definition 5.1. Let G be a graph. An Euler trail of G is a walk $v_0 e_1 v_1 \dots e_k v_k$ such that $\{e_1, \dots, e_k\} = E(G)$ and $e_i \neq e_j \forall i \neq j$. If $v_1 = v_k$ then the walk is a Euler tour.

Theorem 5.3 (Euler)**.** *If G is a connected graph such that the degree of every vertex of G is even then G has an Euler tour.*

Corollary 5.4. *If G is a connected graph such that G contains at most two vertices of odd degree then G has an Euler trail.*

Definition 5.2. A cycle C in G is Hamiltonian if $V(C) = V(G)$.

Lemma 5.5. *Let G be a graph. If there exists $X \subseteq V(G)$, $X \neq \varnothing$ such that $G \setminus X$ has more components than $|X|$ then G has no Hamiltonian cycle.*

Theorem 5.6 (Dirac-Posa)**.** *Let G be a simple graph of $n \geq 3$ vertices. Suppose for every pair of non-adjacent vertices $u, v \in V(G)$, $\deg(u) + \deg(v) \geq n$. Then G has a Hamiltonian cycle.*

Corollary 5.7. *Let G be a simple graph with $n \geq 3$ vertices. Suppose that either:*

1. $\deg(v) \geq \frac{n}{2} \forall v \in V(G)$, *or*

2. $|E(G)| \geq \binom{n}{2} - n + 3$.

Then G has a Hamiltonian cycle.

6 Bipartite Graphs

Definition 6.1. A bipartition of a graph G is a partition (A, B) of $V(G)$ such that every edge of G has exactly one end in A and the other in B.

Definition 6.2. A graph is bipartite if it admits a bipartition.

Lemma 6.1. *Trees are bipartite.*

Theorem 6.2. *Let G be a graph. Then the following are equivalent:*

1. *G is bipartite*

2. *G contains no closed walk of odd length*

3. *G contains no odd cycle (cycle with odd number of vertices)*

7 Matching in Bipartite Graphs

Definition 7.1. A matching M on a graph G is a collection of non-loop edges of G such that every vertex is incident to at most one edge in M. The matching number is the maximum size of a matching in G, denoted $\nu(G)$.

Definition 7.2. $X \subseteq V(G)$ is a vertex cover in G if every edge of G has an end in X. The minimum size of a vertex cover in G is denoted $\tau(G)$.

Lemma 7.1. *Let G be loopless graph. Then $\nu(G) \leq \tau(G) \leq 2\nu(G)$.*

Definition 7.3. Let M be matching in graph G. A path P in G is M-alternating if the edges of P alternate between edges of M and $E(G) \setminus M$ ($\Leftrightarrow$ if every internal vertex of P is incident to an edge of $E(P) \cap M$).

Definition 7.4. An M-alternating path P is M-augmenting if $|V(P)| \geq 2$ and the ends of P are not incident to edges of M.

Lemma 7.2. *A matching M in G has maximum size ($|M| = \nu(G)$) if and only if there does not exist an M-augmenting path in G.*

Theorem 7.3 (Konig)**.** *If G is bipartite then $\nu(G) = \tau(G)$.*

Theorem 7.4. *Let $d \geq 1$ be an integer, let G be bipartite graph such that $\deg_G(v) = d \forall v \in V(G)$. Then G has perfect matching, i.e. every vertex of G is incident to an edge in M.*

Definition 7.5. For a set $S \subseteq V(G)$ let $N(S)$ denote the set of all vertices of G adjacent to at least one vertex in S.

Theorem 7.5 (Hall)**.** *Let G be a bipartite graph with bipartition (A, B). Then G has matching M covering A (i.e. every vertex of A is incident to an edge of M) if and only if $|N(S)| \geq |S|$ for every $S \subseteq A$.*

8 Separations and Menger's Theorem

Definition 8.1. A separation of G is a pair (A, B) such that $A \cup B = V(G)$, no edge of G has one end in $B \setminus A$, the other in $A \setminus B$. The order of separation is $|A \cap B|$.

Remark. *$s, t \in V(G)$ not connected $\Leftrightarrow$ there exists separation (A, B) of order 0 where $s \in A$, $t \in B$.*

Theorem 8.1 (Menger)**.** *Let $s, t \in V(G)$ be a pair of distinct, non-adjacent vertices of G and let $k \geq 1$ be an integer. Then exactly one of the following holds:*

1. *there exists paths $P_1, \ldots, P_k$ in G with ends s, t and otherwise pairwise vertex disjoint;*
2. *there exists a separation (A, B) of G such that $s \in A \setminus B$, $t \in B \setminus A$ of order less than k.*

Theorem 8.2. *Let $Q, R \subseteq V(G)$, $k \geq 1$ integer. Then exactly one of the following holds:*

1. *there exists pairwise disjoint paths $P_1, \ldots, P_k$ in G each with one end in Q, the other in R;*
2. *there exists a separation (A, B) of G of order less than k such that $Q \subseteq A$, $R \subseteq B$.*

Corollary 8.3. *Let G be a k-connected graph, $s, t \in V(G)$ distinct. Then there exist paths $P_1, \ldots, P_k$ from s to t pairwise disjoint except for their ends.*

Definition 8.2. Let $X \subseteq V(G)$, a cut in G corresponding to X, $\delta(X)$, is the collection of all edges of G with one end in X and the other in $V(G) \setminus X$.

Remark. *Every path from $s \in X$ to $t \notin X$ has an edge in $\delta(X)$.*

Definition 8.3. A line graph $L(G)$ of a graph G has vertex set $E(G)$ and $e, f \in V(L(G)) = E(G)$ are adjacent in $L(G)$ if and only if they share an end in G.

Theorem 8.4 (Menger for edge disjoint paths)**.** *Let $s, t \in V(G)$ be distinct. Let $k \geq 1$ be an integer. Then exactly one of the following holds:*

1. *There exists $P_1, \ldots, P_k$ paths in G each with ends s, t such that $E(P_i) \cap E(P_j) = \varnothing$ for $i \neq j$*
2. *There exists $X \subseteq V(G)$ such that $s \in X$, $t \in V(G) \setminus X$, $|\delta(X) < k|$.*

9 Directed Graphs and Network Flows

Definition 9.1. A directed graph or a digraph is a graph in which for every edge e, one of its ends is chosen as the head of e and the other as the tail of e. e is said to be directed from its tail to its head.

Definition 9.2. A directed path from u to v is a path from u to v in which every edge is traversed from its tail to its head as we follow the path from u to v.

Definition 9.3. For $X \subseteq V(G)$ let $\delta^+(X)$ be the set of all edges of G with tail in X and head in $V(G) \setminus X$. Let $\delta^-(X)$= $\delta^+(V(G) \setminus X)$. For $v \in V(G)$ let $\delta^+(v)$=$\delta^+(\{v\})$ and $\delta^-(v)$=$\delta^-(\{v\})$.

Lemma 9.1. *Let G be a digraph. Let $s, t \in V(G)$. Then there does not exist a directed path in G from s to t if and only if there eixsts $X \subseteq V(G)$ such that $s \in X$, $t \in V(t) \setminus X$, $\delta^+(X) = \varnothing$.*

Definition 9.4. Let G be a digraph, $s, t \in V(G)$ distinct. A function $\phi : E(G) \to \mathbb{R}^+$ is an (s, t)-flow on G if

$$\sum_{e \in \delta^-(v)} \phi(e) = \sum_{e \in \delta^+(v)} \phi(e)$$

for every $v \in V(G) \setminus \{s,t\}$. The value of ϕ is

$$\sum_{e \in \delta^-(s)} \phi(e) - \sum_{e \in \delta^+(s)} \phi(e)$$

Lemma 9.2. *Let ϕ be an (s,t)-flow on a digraph G with value k. Then for any $X \subseteq V(G)$ such that $s \in X$, $t \in V(G) \setminus X$, we have*

$$\sum_{e \in \delta^+(X)} \phi(e) - \sum_{e \in \delta^-(X)} \phi(e) = k$$

Definition 9.5. A flow $\phi : E(G) \to \mathbb{R}_+$ is integral if $\phi(e) \in \mathbb{Z}_+$ for every $e \in E(G)$.

Lemma 9.3. *Let ϕ be an integral (s,t)-flow on a digraph G with value $k \geq 0$. Then there exist directed paths $P_1, \ldots, P_k$ from s to t such that every edge of G belongs to at most $\phi(e)$ of these paths.*

Definition 9.6. Let $c : E(G) \to \mathbb{Z}_+$ be a capacity function. An (s,t)-flow ϕ is c-admissible if $\phi(e) \leq c(e)$ for every $e \in E(G)$.

Definition 9.7. Given graph G and capacity function c, a path P in G from s to v is ϕ-augmenting for an (s,t)-flow ϕ if

- $\phi(e) \leq c(e) - 1$ if $e \in E(P)$ is traversed in the forward direction as we go from s to v along P, and
- $\phi(e) \geq 1$ if $e \in E(P)$ is traversed in the backward direction.

Lemma 9.4. *Let ϕ be an integral c-admissible (s,t)-flow on G of value k. If there exists a ϕ-augmenting path P from s to t then there exists an integral c-admissible (s,t)-flow on G of value $k+1$.*

Theorem 9.5 (Max flow min cut, Ford-Fulkerson)**.** *Let $k \geq 1$ be an integer and let c be a capacity function. Then exactly one of the following holds:*

1. *There exists an integral c-admissible (s,t)-flow of value at least k*
2. *There exists $X \subseteq V(G)$, $s \in X$, $t \notin X$ such that*

$$\sum_{e \in \delta^+(X)} c(e) < k$$

10 Independent Sets, Cliques and Ramsey Theorem

Definition 10.1. A set $S \subseteq V(G)$ is independent if no edge of G has both ends in S. $\alpha(G)$, the independence number, is the maximum size of an independent set.

Remark. *No $v \in S$ independent set can be incident to a loop.*

Definition 10.2. A set $L \subseteq E(G)$ is an edge covering of G if every vertex of G is incident to an edge of L. $\rho(G)$ is the minimum size of an edge covering in G (only well-defined if every vertex of G is incident to at least one edge).

Remark. *$\rho(G) \geq \alpha(G)$ and $\rho(G) \geq \frac{|V(G)|}{2}$*

Lemma 10.1. *$\alpha(G) + \tau(G) = |V(G)|$ for any graph G.*

Theorem 10.2 (Gallai)**.** *Let G be a simple graph such that every vertex of G is incident to an edge. Then $\nu(G) + \rho(G) = |V(G)|$.*

Corollary 10.3. *Let G be a simple bipartite graph such that every vertex is incident to an edge. Then $\alpha(G) = \rho(G)$.*

Definition 10.3. Let G be a simple graph. The complement of G is the graph $\overline{G}$ such that $V(G) = V(\overline{G})$ and a pair of vertices is adjacent in G if and only if it is non-adjacent in $\overline{G}$.

Definition 10.4. A clique $X \subseteq V(G)$ is a set of pairwise adjacent vertices. $\omega(G)$, the clique number, is the maximum size of a clique in G, or, equivalently, the maximum t such that K_t is a subgraph of G

Remark. *If G is simple then X is a clique in $G \Leftrightarrow X$ is independent in $\overline{G}$.*

Definition 10.5. Given integer $s, t \geq 1$, the Ramsey number $R(s,t)$ is the minimal N such that every simple graph G with $|V(G)| = N$ either contains an independent set of size s or a clique of size t (or both).

Remark. *$R(s,t) = R(t,s)$, $R(1,t) = 1$ and $R(2,t) = t$.*

Theorem 10.4 (Ramsey, Erdos-Szekeres)**.** *$R(s,t)$ exists for all $s, t \geq 1$ and*

$$R(s,t) \leq R(s-1,t) + R(s,t-1)$$

for $s, t \geq 2$.

Corollary 10.5. *For $s, t \geq 1$,*

$$R(s,t) \leq \binom{s+t-2}{s-1}$$

Lemma 10.6. *If*

$$\binom{N}{s} 2^{1-\binom{s}{2}} < 1$$

then there exists a simple graph G with $|V(G)| = N$ and no clique or independent set of size s (i.e. $R(s,s) > N$).

Theorem 10.7 (Erdos)**.** *For $s \geq 2$, $R(s,s) \geq 2^{\frac{s}{2}} = (\sqrt{2})^s$.*

11 Vertex Coloring

Definition 11.1. Let G be a graph and S a set with $|S| = k$. We say that $c : V(G) \to S$ is a (proper) k-coloring of G if for every $e \in E(G)$ with ends u, v we have $c(u) \neq c(v)$.

Definition 11.2. The chromatic number $\chi(G)$ of a graph G is the minimum k such that there exists a k-coloring of G. If G has a loop then no k-coloring of G is possible, so $\chi(G) = \infty$.

Remark. *G is 1-colorable $\Leftrightarrow$ G is edgeless; G is 2-colorable $\Leftrightarrow$ G is bipartite.*

Definition 11.3. The set S in the definition of k-coloring is the set of colors. The set of all vertices of a given color is the color class of that color (formally $\{v \in V(G) : c(v) = s\}$ for some $s \in S$).

Lemma 11.1. *Let G be a loopless graph. Then*

$$\chi(G) \geq \omega(G)$$

and

$$\chi(G) \geq \left\lceil \frac{|V(G)|}{\alpha(G)} \right\rceil$$

Definition 11.4. A graph G is k-degenerate if every non-null subgraph of G contains a vertex of degree in the subgraph at most k (i.e. for every $H \subseteq G$ non-null there exists $v \in V(H) : \deg_H(v) \leq k$).

Remark. *G is 1-degenerate $\Leftrightarrow$ G is a forest.*

Lemma 11.2. *If G is loopless and k-degenerate then $\chi(G) \leq k + 1$.*

Definition 11.5. $\underline{\Delta(G)}$ denotes the maximum degree of a vertex in G.

Remark. *Every graph is $\Delta(G)$-degenerate.*

Corollary 11.3. *If G is loopless then $\chi(G) \leq \Delta(G) + 1$*

Theorem 11.4 (Brooks)**.** *Let G be a connected loopless graph such that G is not complete and G is not an odd cycle. Then $\chi(G) \leq \Delta(G)$.*

12 Edge Coloring

Definition 12.1. Let G be a loopless graph. $c : E(G) \to S$ with $|S| = k$ is a k-edge coloring of G if $c(e) \neq c(f)$ for any pair of distinct $e, f \in E(G)$ such that e, f share an end. The edge coloring number (or edge chromatic number) $\underline{\chi'(G)}$ is the minimum k such that G admits a k-edge coloring.

Lemma 12.1.

$$\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$$

for any loopless graph G with $\Delta(G) \geq 1$.

Definition 12.2. A graph G is k-regular if $\deg_G(v) = k$ for every $v \in V(G)$.

Lemma 12.2. *Let G be a graph with $\Delta(G) \leq k$. Then there exists a k-regular graph H such that G is a subgraph of H. Moreover, if G is loopless (resp. bipartite, simple) then H be can be chosen to be loopless (resp. bipartite, simple).*

Theorem 12.3 (Konig)**.** *If G is bipartite then $\chi'(G) = \Delta(G)$.*

Definition 12.3. A 2-factor in a loopless graph G is a $F \subseteq E(G)$ such that every vertex of G is incident to exactly 2 edges of F.

Lemma 12.4. *Let G be a loopless $2k$-regular graph. Then $E(G)$ can be partitioned in k 2-factors.*

Theorem 12.5 (Shannon)**.** *Let G be a loopless graph. Then $\chi'(G) \leq 3\left\lceil \frac{\Delta(G)}{2} \right\rceil$.*

Remark. *If G simple then a stronger result exists: $\chi'(G) \leq \Delta(G) + 1$ by Vizing.*

13 Graph Minors and Hadwiger's Conjecture

Definition 13.1. Let e be a non-loop edge of G with ends u and v. We say that G' is a graph obtained from G by contracting e if G' is obtained by deleting e and identifying u, v to a single vertex, called a new vertex.

Definition 13.2. A graph H is a minor of G if H can be obtained from G by repeatedly deleting vertices and/or deleting edges and/or contracting edges.

Remark. *Every graph is a minor of itself and the minor relation is transitive: if J is a minor of H and H a minor of G then J is a minor of G.*

Remark. *A graph has no K_2 minor $\Leftrightarrow$ it has no K_2 subgraph $\Leftrightarrow$ all edges are loops. A graph has no C_1 minor $\Leftrightarrow$ it is a forest. A graph has no K_3 minor $\Leftrightarrow$ it has no cycle of length 3 or more $\Leftrightarrow$ it is a forest with added loops and parallel edges.*

Definition 13.3. A graph G is a subdivision of a graph H if G is obtained from H by replacing edges by internally vertex disjoint paths (i.e. by replacing $e \in E(H)$ with ends u, v by paths $P_1, \ldots, P_k$ from u to v vertex disjoint except at the ends).

Remark. *If G is a subdivision of H then H (or a graph isomorphic to H) is a minor of G.*

Lemma 13.1. *If G is 3-connected then G has a K_4 minor.*

Lemma 13.2. *Let G be a simple graph with no K_4 minor. Let X be a clique in G with $|X| \leq 2$ and $X \neq V(G)$. Then there exists $v \in V(G) \setminus X$ such that $\deg_G(v) \leq 2$.*

Theorem 13.3 (Hadwiger's conjecture for $t = 3$). *If G is a loopless graph with no K_4 minor then $\chi(G) \leq 3$.*

14 Planar Graphs

Definition 14.1. A (planar) drawing of a graph G in the plane represents vertices of G as distinct points in the plane $\mathbb{R}^2$ and edges of G as curves which join the points corresponding to their ends, such that these curves do not intersect themselves or each other.

Definition 14.2. A graph G is planar if it admits a planar drawing.

Definition 14.3. The points of the plane which do not belong to the drawing of G are divided into regions, where two points belong to the same region if they can be joined by a curve which does not intersect the drawing.

Remark. *The Jordan curve theorem states that any closed simple curve (a continuous injective function $\phi : [0, 1] \to \mathbb{R}^2$) separates the plane into two regions.*

Lemma 14.1. *Let G be a graph drawn in the plane. Let $e \in E(G)$. Then the regions on different sides of e are the same if and only if e is a cut-edge of G.*

Definition 14.4. Given a planar graph G, let $\mathrm{Reg}(G)$ denote the number of regions in any drawing of G in the plane.

Theorem 14.2 (Euler's formula)**.** *Let G be a planar non-null graph. Then*

$$|V(G)| - |E(G)| + Reg(G) = 1 + comp(G)$$

Remark. *$Reg(G)$ is independent on the drawing. If G is connected then $|V(G)| - |E(G)| + Reg(G) = 2$.*

Definition 14.5. The length of a region of a drawing of G is the number of edges on its boundary, with edges such that this region lies on both sides of them counted twice.

Lemma 14.3. *Let G be a connected simple graph drawn in the plane, with $|E(G)| \geq 2$. Then the length of every region of G is at least 3, and if it is 3 then the boundary is a cycle of length 3.*

Lemma 14.4. *If G is a simple planar graph, $|E(G)| \geq 2$ then $|E(G)| \leq 3|V(G)| - 6$. If G contains no length 3 cycles then $|E(G)| \leq 2|V(G)| - 4$.*

Definition 14.6. $K_{m,n}$ called complete bipartite graph is a simple bipartite graph that admits a bipartition (A, B) with $|A| = m$, $|B| = n$ and every vertex of A is adjacent to every vertex of B.

Remark. *$|E(K_{m,n})| = mn$; $|E(K_{3,3})| = 9 > 2|V(K_{3,3})| - 4 = 8$ so $K_{3,3}$ is non planar.*

Corollary 14.5. *Let G be a simple palanr graph, $|E(G)| \geq 2$. Then*

$$\sum_{v \in V(G)} (6 - \deg(v)) \geq 12$$

Corollary 14.6. *If G is a simple non-null planar graph then $\deg_G(v) \leq 5$ for some $v \in V(G)$ (thus G is 5-degenerate and $\chi(G) \leq 6$).*

15 Kuratowski's Theorem

Lemma 15.1. *Let G be a 2-connected loopless graph drawn in the plane. Then every region is bounded by a cycle.*

Lemma 15.2. *Let C be a cycle, $X, Y \subseteq V(C)$, $|V(C)| \geq 2$. Then at least one of the following holds:*

1. *There exist $z_1, z_2 \in V(C)$ distinct and two paths P, Q with ends z_1 and z_2 such that $P \cup Q = C$, $X \subseteq V(P)$, $Y \subseteq V(Q)$*
2. *There exists distinct $x_1, x_2 \in X$, $y_1, y_2 \in Y$ such that x_1, y_1, x_2, y_2 appear on C in this order*
3. *$X = Y$ and $|X| = |Y| = 3$.*

Theorem 15.3 (Kuratowski-Wagner)**.** *A graph G is non-planar if and only if either K_5 or $K_{3,3}$ is a minor of G.*

Theorem 15.4 (Kuratowski)**.** *A graph G is non-planar if and only if G contains a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

Remark. *There is a theorem that extends Kuratowski's theorem to the projective plane due to Archdeacon: there is a list of 35 graphs such that a graph G can be drawn in the projective plane if and only if it contains none of them as minors (equivalently, if G does not contain as subgraph a subvision of one of 103 graphs).*

Remark. *There is a theorem due to Robertson and Seymour that states for any surface Σ there exists a finite list $H_1, \ldots, H_k$ of graphs such that G can be drawn on Σ if and only if it contains no H_i as minor.*

16 The Four Color Theorem

Theorem 16.1 (Heawood)**.** *If G is planar and loopless then $\chi(G) \leq 5$.*

Definition 16.1. A drawing of G in the plane is a triangulation if the boundary of every region is a triangle (cycle of length 3).

Remark. *Maximal planar simple graphs correspond to triangulations.*

Definition 16.2. Let G be a connected graph drawn in the plane. The graph G^* drawn in the plane is the dual of G if

- Every region of G contains exactly one vertex of G^*,
- every edge of G is crosed by exactly one of G^* and the drawings of G and G^* are otherwise disjoint, and
- $|E(G)| = |E(G^*)|$

Theorem 16.2 (Tait)**.** *Let G be a planar triangulation and let G^* be its dual. Then $\chi(G) \leq 4 \Leftrightarrow \chi'(G^*) = 3$.*

Remark. *This shows that the four color theorem is equivalent to the statement "every 3-regular 2-connected planar graph is 3-edge colorable".*

Remark. *Consider this theorem due to Kaufman: for any pair of "bracketings" of the product $u_1 \times \cdots \times u_m$ there exists a choice of $u_n \in \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ for every $1 \leq n \leq m$ such that the corresponding products are the same and non-zero. This theorem is equivalent to the four color theorem.*