Introduction

Definition 0.1 (Riemann 1854)**.** Let $[a,b]$ be a closed bounded interval, $f:[a,b]\to\mathbb{R}$ bounded function. We say f is *Riemann integrable* if

$$\begin{aligned}\underline{\int_a^b} f &:= \sup\{\sum_{i=1}^n \inf_{[x_{i-1},x_i]} f(x_i - x_{i-1}) : a = x_0 < x_1 < \cdots < x_n = b\}\\ = \overline{\int_a^b} f &:= \inf\{\sum_{i=1}^n \sup_{[x_{i-1},x_i]} f(x_i - x_{i-1}) : a = x_0 < x_1 < \cdots < x_n = b\}\end{aligned}$$

We then denote $\int_a^b f = \int_a^b f(x)dx := \underline{\int_a^b} f = \overline{\int_a^b} f$.

Theorem 0.1. *Every continuous function $f:[a,b]\to\mathbb{R}$ is Riemann integrable*

Remark. $f : x \in [0,1] \mapsto \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}\end{cases}$ is *not* Riemann integrable.

1 Measure Theory

Definition 1.1.

1. Let rectangle R be $(a_1,b_1)\times\cdots\times(a_d,b_d) \subseteq R \subseteq [a_1,b_1]\times\cdots\times[a_d,b_d]$, where $-\infty < a_i \le b_i < \infty \forall 1 \le i \le d$. We call *volume* of R and denote $\mathrm{vol}(R)$ the number $\mathrm{vol}(R) := \prod_{i=1}^d (b_i - a_i)$. We say that R is a *cube* if $b_1 - a_1 = \cdots = b_d - a_d$.

2. For every set $A \subseteq \mathbb{R}^d$ we call the *exterior measure* of A and denote $m_*(A)$ the number

$$m_*(A) = \inf\left\{\sum_{k=1}^\infty \mathrm{vol}(Q_k) : Q_k \text{ closed cubes}, A \subseteq \bigcup_{k=1}^\infty Q_k\right\} \in [0,\infty]$$

Remark.

$$\left\{\sum_{k=1}^\infty \mathrm{vol}(Q_k) : Q_k \text{ closed cubes}, A \subseteq \bigcup_{k=1}^\infty Q_k\right\} \neq \varnothing \because A \subseteq \bigcup_{n=1}^\infty [-n,n]^d = \mathbb{R}^d$$

Remark.

$$\begin{aligned} m_*(A) &= \inf\left\{\sum_{k=1}^{\infty}\mathrm{vol}(Q_k) : Q_k \text{ open cubes}, A \subseteq \bigcup_{k=1}^{\infty} Q_k\right\} \\ &= \inf\left\{\sum_{k=1}^{\infty}\mathrm{vol}(Q_k) : Q_k \text{ rectangles}, A \subseteq \bigcup_{k=1}^{\infty} Q_k\right\} \end{aligned}$$

Proposition 1.1. *If $A \subseteq \mathbb{R}^d$ is countable then $m_*(A) = 0$*

Proposition 1.2 (monotonicity)**.** *If $A \subseteq B \subseteq \mathbb{R}^d$ then $m_*(A) \leq m_*(B)$*

Proposition 1.3. *If $O \subseteq \mathbb{R}^d$ is open then it can be written as $O = \bigcup_{k=1}^{\infty} \overline{Q}_k$ where Q_k are disjoint, open cubes ($\overline{Q_k}$ is the closure of Q_k).*

Proposition 1.4. *If $R \subseteq \mathbb{R}^d$ is a rectangle then $m_*(R) = \mathrm{vol}(R)$.*

Proposition 1.5. *If $A \subseteq \mathbb{R}^d$ then $m_*(A) = \inf\{m_*(O) : O \text{ open set}, A \subseteq O\}$.*

Proposition 1.6. *Let $(A_k)_{k\in\mathbb{N}}$ be a sequence of sets in $\mathbb{R}^d$ (not necessarily disjoint). Then $m_*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} m_*(A_k)$.*

Proposition 1.7. *Let $A_1, A_2 \subseteq \mathbb{R}^d$ be such that $d(A_1, A_2) > 0$ i.e. $\inf\{|x - y| : x \in A_1, y \in A_2\} > 0$. Then $m_*(A_1 \cup A_2) = m_*(A_1) + m_*(A_2)$*

Definition 1.2. A set $A \subseteq \mathbb{R}^d$ is said to be *(Lebesgue)-measurable* if for every $\epsilon > 0$, there exists O_ϵ open such that $A \subseteq O_\epsilon$ and $m_*(O_\epsilon \setminus A) < \epsilon$. We then denote $m(A) = m_*(A)$ the *(Lebesgue)-measure* of A.

Proposition 1.8.

1. *If $m_*(A) = 0$ then A is measurable.*
2. *A countable union of measurable sets is measurable.*
3. *Open sets and closed sets are measurable.*
4. *If A is measurable then $R^d \setminus A =: A^c$ is measurable.*
5. *A countable intersection of measurable sets is measurable.*

Theorem 1.9 (countable additivity)**.** *Let $(A_k)_{k\in\mathbb{N}}$ be measurable and disjoint. Then*

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k)$$

Remark. In particular, if $A \subseteq B \subseteq \mathbb{R}^d$ are measurable then $m(B) = m(A) + m(B \setminus A)$.

Proposition 1.10 (continuity of measure)**.** *Let $(A_k)_{k\in\mathbb{N}}$ be measurable.*

1. *If $A_k \subseteq A_{k+1} \forall k \in \mathbb{N}$ then $m(\bigcup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} m(A_k)$.*
2. *If $A_k \supseteq A_{k+1} \forall k \in \mathbb{N}$ and $m(A_1) < \infty$ then $m(\bigcap_{k=1}^{\infty} A_k) = \lim_{k\to\infty} m(A_k)$.*

   Remark. $m(A_1) < \infty$ is necessary: $m(\bigcap_{k=1}^{\infty}[k,\infty)) = m(\varnothing) = 0$ while $m([k,\infty)) = \infty \forall k \in \mathbb{N}$.

Theorem 1.11 (outer and inner approximations of measurable sets)**.** *Let $A \subseteq \mathbb{R}^d$. Then the following are equivalent:*

1. *A is measurable;*
2. *There exists a G_δ set G (a G_δ set is a countable intersection of open sets) and a set N of measure 0 such that $A = G \setminus N$;*
3. *For every $\epsilon > 0$, there exists F_ϵ closed such that $F_\epsilon \subseteq A$ and $m_*(A \setminus F_\epsilon) < \epsilon$;*
4. *There exists an F_σ set F (an F_σ set is a countable union of closed sets) and a set N of measure 0 such that $A = F \cup N$.*

Counterexamples

Are all subsets of R^d measurable?

Theorem 1.12. *If $A \subseteq \mathbb{R}^d$ is such that $m_*(A) > 0$ then there exists $B \subseteq A$ non-measurable.*

Are all subsets of measure 0 in R countable?

Definition 1.3. We call *Cantor set* the set $C := \bigcap_{k=1}^{\infty} C_k$ where $C_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $\forall k \geq 2$, $C_k := \bigcup_{j=1}^{2^k} I_{j,k}$ where $\forall j \in \{1, \ldots, 2^{k-1}\}$, $I_{2j-1,k}, I_{2j,k}$ are the first and last thirds of $I_{j,k-1}$.

Theorem 1.13. *C is closed and uncountable. $m(C) = 0$.*

Are all measurable sets Borel?

Definition 1.4. A collection Ω of subsets of $\mathbb{R}^d$ is called a *σ-algebra* if the following conditions are satisfied:

1. $\mathbb{R}^d \in \Omega$;
2. $\forall A, B \in \Omega : A \setminus B \in \Omega$;
3. $\forall (A_k)_{k\in\mathbb{N}} \subseteq \Omega : \bigcup_{k=1}^{\infty} A_k \in \Omega$.

Proposition 1.14. *Any intersection of σ-algebras is a σ-algebra.*

Definition 1.5. The intersection of all the σ-algebras containing the open sets is called the *Borel σ-algebra* and its elements the *Borel sets*.

Remark. In particular, Borel sets are measurable.

Proposition 1.15. *There exists a subset of the Cantor set which is measurable but not Borel.*

Definition 1.6. We call *Cantor-Lebesgue function* (or *Cantor staircase function*) the function

$$\begin{aligned}
&\varphi : [0,1] \to [0,1], \\
&\varphi(x) = \frac{i}{2^k} \text{ if } x \in J_{k,i} \text{ where } J_{k,i} \text{ is the } i\text{-th interval of } [0,1] \setminus C_k, k \geq 1, i \in \{1, \ldots, 2^k - 1\}, \\
&\varphi(0) = 0, \varphi(x) = \sup\{\varphi(y) : y \in [0,x) \setminus C\} \text{ if } x \in (0,1] \cap C
\end{aligned}$$

Remark. $\varphi(1) = 1$.

Proposition 1.16. $\varphi : [0,1] \to [0,1]$ *is increasing, continuous and surjective.*

Proposition 1.17. *If $D \subseteq \mathbb{R}$ is not Borel, then $D \times \{0\}^{d-1} \subseteq \mathbb{R}^d$ is not Borel.*

2 Lebesgue Measurable Function

Remark. We denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

Proposition 2.1. *Let $A \subseteq \mathbb{R}^d$ be measurable, $f : A \to \overline{\mathbb{R}}$. Then the following are equivalent:*

1. *$\forall c \in \mathbb{R} : f^{-1}((c, +\infty])$ is measurable*

2. *$\forall c \in \mathbb{R} : f^{-1}([c, +\infty])$ is measurable*

3. *$\forall c \in \mathbb{R} : f^{-1}([-\infty, c))$ is measurable*

4. *$\forall c \in \mathbb{R} : f^{-1}([-\infty, c])$ is measurable*

Definition 2.1. When these are satisfied, we say f is *(Lebesgue) measurable.*

Proposition 2.2. *Let $A \subseteq \mathbb{R}^d$ and $(A_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^d$ be measurable sets such that the sets $(A_k)_{k\in\mathbb{N}}$ are disjoint and $\bigsqcup_{k=1}^{\infty} A_k = A$. Let $f : A \to \overline{\mathbb{R}}$ be a function. If $f|_{A_k}$ is measurable for all $k \in \mathbb{N}$ then f is measurable.*

Proposition 2.3. *Let $A \subseteq \mathbb{R}^d$ measurable.*

1. *$\forall B \subseteq A$ measurable, $\forall f : A \to \overline{\mathbb{R}}$ measurable, $f|_B$ is measurable;*

2. *$\forall B \subseteq \mathbb{R}$ Borel, $\forall f : B \to \mathbb{R}$ continuous, $\forall g : A \to B$ measurable, then $f \circ g$ is measurable;*

3. *$\forall f : A \to \overline{\mathbb{R}}$, $\forall g : A \to \mathbb{R}$ both measurable, $f + g$ is measurable;*

4. *$\forall f : A \to [0, \infty]$ measurable, $\forall k \in \mathbb{N}$, f^k is measurable;*

5. *$\forall f, g : A \to \mathbb{R}$ measurable, $f \cdot g$ is measurable;*

6. *$\forall f, g : A \to \mathbb{R}$ measurable, $\max(f, g), \min(f, g)$ is measurable.*

Proposition 2.4. *Let $A \subseteq \mathbb{R}^d$ be measurable, let $f : A \to \overline{\mathbb{R}}$ measurable. Then for every Borel set $B \subseteq \mathbb{R}$, $f^{-1}(B)$ is measurable.*

Remark. $\exists D \subseteq \mathbb{R}$ measurable, f measurable (even continuous) such that $f^{-1}(D)$ is not measurable.

Proposition 2.5. *Let $A \subseteq \mathbb{R}^d$ measurable, $f : A \to \mathbb{R}$ continuous, then f is measurable.*

Definition 2.2. Let $A \subseteq \mathbb{R}^d$, $P(x)$ a statement depending on $x \in A$. We say $P(x)$ is *true for almost every $x \in A$* (or *a.e.* $x \in A$) if $m_*(\{x \in A : P(x) \text{ is false}\}) = 0$.

Proposition 2.6. *If $(P_k(x))_{k\in\mathbb{N}}$ is a countable collection of statements depending on $x \in A$, then*

$$\left[\forall k \in \mathbb{N} : \text{ for a.e. } x \in A, P_k(x) \text{ is true}\right] \Leftrightarrow \left[\text{for a.e. } x \in A, \forall k \in \mathbb{N} : P_k(x) \text{ is true}\right]$$

Proposition 2.7. *Let $f, g : A \to \overline{\mathbb{R}}$ be such that $f = g$ a.e. in A. Then f measurable if and only if g measurable.*

Proposition 2.8. *Let $A \subseteq \mathbb{R}^d$ and $(A_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^d$ be measurable sets such that the sets $(A_k)_{k\in\mathbb{N}}$ disjoint and $\bigcup_{k=1}^{\infty} A_k = A$. Let $f : A \to \overline{\mathbb{R}}$ be a function. If $f|_{A_k}$ is measurable for all $k \in \mathbb{N}$, then f is measurable.*

Proposition 2.9. *Let $A \subseteq \mathbb{R}^d$ be measurable*

1. *$\forall B \subseteq A$ measurable, $\forall f : A \to \overline{\mathbb{R}}$ measurable, $f|_B$ is measurable.*
2. *$\forall B \subseteq \mathbb{R}$ Borel, $\forall f : B \to \mathbb{R}$ continuous, $\forall g : A \to B$ measurable, $f \circ g$ is measurable.*
3. *$\forall f : A \to \overline{\mathbb{R}}$ measurable, $\forall g : A \to \mathbb{R}$ measurable, $f + g$ is measurable.*
4. *$\forall f : A \to \overline{\mathbb{R}}$ measurable, $\forall k \in \mathbb{N}$, f^k is measurable.*
5. *$\forall f, g : A \to \mathbb{R}$, $f \cdot g$ is measurable.*

Remark. $\exists f, g$ measurable such that $f \circ g$ is not measurable.

Proposition 2.10. *Let $(f_n)_{n\in\mathbb{N}}, f_n : A \to \overline{\mathbb{R}}$ be measurable functions converging pointwise a.e. in A to a function $f : A \to \overline{\mathbb{R}}$ i.e. $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x \in A$. Then f is measurable.*

Proposition 2.11. *Let $(f_n)_{n\in\mathbb{N}}, f_n : A \to \overline{\mathbb{R}}$ be measurable functions. Then*

$$x \mapsto \inf_{n\in\mathbb{N}} f_n(x), x \mapsto \sup_{n\in\mathbb{N}} f_n(x), x \mapsto \liminf_{n\to\infty} f_n(x), x \mapsto \limsup_{n\to\infty} f_n(x)$$

are all measurable.

Definition 2.3. We call *simple function* a measurable function $\varphi : A \to \mathbb{R}$ such that $\varphi(A)$ is finite and φ has finite support i.e. $m(\{x \in A : \varphi(x) \neq 0\}) < \infty$.

Remark. In particular, any simple function φ can be written as

$$\varphi = \sum_{i=1}^{n} c_i \chi_{A_i}$$

where $n \geq 0$, $c_1, \dots, c_n \in \mathbb{R} \setminus \{0\}$ distinct (such that $\varphi(A) \setminus \{0\} = \{c_1, \dots, c_n\}$) and $A_1, \dots, A_n \subseteq A$ measurable, disjoint and with finite measure ($A_i = \varphi^{-1}(\{c_i\})$).

Definition 2.4. We say that $\sum_{i=1}^{n} c_i\chi_{A_i}$ is the *canonical form* of the simple function φ. We say that $\sum_{i=1}^{n} c_i\chi_{A_i}$ is a *step function* if the A_i are rectangles.

Theorem 2.12 (Simple Approximation Lemma)**.** *Let $f : A \to \mathbb{R}, m(A) < \infty$ be measurable and bounded i.e. $\exists M > 0 \forall x \in A : |f(x)| < M$. Then $\forall \epsilon > 0 \exists \varphi_\epsilon, \psi : A \to \mathbb{R}$ simple functions such that*

$$\varphi_\epsilon \leq f \leq \psi_\epsilon < \varphi_\epsilon + \epsilon$$

Theorem 2.13 (Simple Approximation Theorem)**.** *Let $f : A \to \overline{\mathbb{R}}$ be measurable. Then $\exists (\varphi_n)_{n\in\mathbb{N}}$ simple functions such that*

1. *$(\varphi_n)_{n\in\mathbb{N}}$ converges pointwise to f on A i.e. $\lim_{n\to\infty} \varphi_n(x) = f(x) \forall x \in A$, and*

2. *$|\varphi_n| \leq |\varphi_{n+1}| \leq |f|$ on A $\forall n \in \mathbb{N}$.*

If $f \geq 0$ then we have moreover that $\varphi_n \geq 0$ and increasing in n.

Theorem 2.14 (Egorov)**.** *Let $A \subseteq \mathbb{R}^d$ be measurable, $m(A) < \infty$, $(f_n)_{n\in\mathbb{N}}, f_n : A \to \mathbb{R}$ measurable converging pointwise to $f : A \to \mathbb{R}$. Then $\forall \epsilon > 0 \exists F_\epsilon \subseteq A$ closed such that $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly in F_ϵ i.e.*

$$\sup_{F_\epsilon} |f_n - f| \xrightarrow{n\to\infty} 0$$

and $m(A \setminus F_\epsilon) < \epsilon$

Remark. This result does not hold in general when $m(A) = \infty$, e.g. $f_n(x) = \frac{x}{n}$ on $A = \mathbb{R}$.

Theorem 2.15 (Lusin)**.** *Let $f : A \to \mathbb{R}$ be measurable. Then $\forall \epsilon > 0 \exists F_\epsilon \subseteq A$ closed such that $f|_{F_\epsilon}$ is continuous on F_ϵ and $m(A \setminus F_\epsilon) < \epsilon$.*

Remark. Recall that "$f|_F$ continuoous on F" $\neq$ "f continuous on F": $\chi_{\mathbb{Q}}$ is not continuous at any point in $\mathbb{R}$ but $\chi_{\mathbb{Q}}|_{\mathbb{R}\setminus\mathbb{Q}} = 0$ is continuous on $\mathbb{R} \setminus \mathbb{Q}$.

3 Lebesgue Integration

Case of a simple function

Definition 3.1. Let $\varphi : A \to \mathbb{R}$ be a simple function and $\varphi = \sum_{k=1}^{n} c_k \chi_{A_k}$ be its canonical form. We define the *(Lebesgue) integral* of φ over A by

$$\int_A \varphi = \int_A \varphi(x)dx = \sum_{k=1}^{n} c_k m(A_k)$$

For any $B \subseteq A$ measurable, we define $\int_B f = \int_A f \chi_B$.

Proposition 3.1 (independence of the representation). *Let $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{R}$ and $A_1, \ldots, A_n \subseteq A$ be measurable, disjoint, $m(A_k) < \infty$. Then*

$$\int_A \sum_{k=1}^{n} c_k \chi_{A_k} = \sum_{k=1}^{n} c_k m(A_k)$$

Proposition 3.2. *Let $\varphi, \psi : A \to \mathbb{R}$ be simple. Then*

1. *$\forall \alpha, \beta \in \mathbb{R}$, $\alpha\varphi + \beta\psi$ simple and $\int_A (\alpha\varphi + \beta\psi) = \alpha \int_A \varphi + \beta \int_A \psi$.*
2. *$\forall B_1, B_2 \subseteq A$ measurable disjoint, $\int_{B_1 \cup B_2} \varphi = \int_{B_1} \varphi + \int_{B_2} \varphi$.*
3. *If $\varphi \leq \psi$ on A then $\int_A \varphi \leq \int_A \psi$.*
4. *$|\varphi|$ is simple and $|\int_A \varphi| \leq \int_A |\varphi|$.*

Case of a bounded measurable function with finite support

Definition 3.2. We denote $\operatorname{supp}(f)$ and call *support* of a measurable function $f : A \to \overline{R}$ the set

$$\operatorname{supp}(f) = \{x \in A : f(x) \neq 0\}$$

If $\operatorname{supp}(f) \subseteq E \subseteq A$, then we say f is *supported in E*. If $m(\operatorname{supp}(f)) < \infty$, we say that f has *finite support.*

Proposition 3.3. *Let $f : A \to \mathbb{R}$ be bounded, measurable and with finite support. Let $(\varphi_n)_{n\in\mathbb{N}}$ be simple functions in A such that*

1. *$\exists E \subseteq A$ measurable such that $m(E) < \infty$ and supp$(\phi_n) \subseteq E$ for all $n \in \mathbb{N}$,*

2. *$\exists M > 0$ such that $\forall n \in \mathbb{N}$, $|\varphi_n| \leq M$ in A, and*

3. *$\lim_{n\to\infty} \varphi_n(x) = f(x)$ for a.e. $x \in A$ (a.e. pointwise convergence).*

Then $\lim_{n\to\infty} \int_A \varphi_n$ exists and does not depend on the choice of $(\varphi_n)_{n\in\mathbb{N}}$ satisfying the above.

Remark. Such (φ_n) exists by the Simple Approximation Lemma.

Definition 3.3. Given the above proposition, we then call *integral of f over A* the number $\int_A f = \lim_{n\to\infty} \int_A \varphi_n$. For every $B \subseteq A$ measurable, we define $\int_B f = \int_A f\chi_B$.

Remark. If $f = 0$ a.e. in A then $\int_A f = 0$.

Proposition 3.4. *Let $f, g : A \to \mathbb{R}$ be bounded, measurable and with finite support. Then*

1. *$\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is bounded, measurable and with finite support and $\int_A(\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$.*

2. *$\forall B_1, B_2 \subseteq A$ measurable disjoint, $\int_{B_1\cup B_2} f = \int_{B_1} f + \int_{B_2} f$.*

3. *If $f \leq g$ on A then $\int_A f \leq \int_A g$.*

4. *$|f|$ is bounded, measurable and with finite support and $|\int_A f| \leq \int_A |f|$.*

Theorem 3.5 (Bounded Convergence Theorem)**.** *Let $(f_n)_{n\in\mathbb{N}}, f_n : A \to \mathbb{R}$ be a sequence of measurable function such that*

1. *$\exists E \subseteq A$ measurable such that $m(E) < \infty$ and supp$(f_n) \subseteq E$ for all $n \in \mathbb{N}$,*

2. *$\exists M > 0$ such that $\forall n \in \mathbb{N}$, $|f_n| \leq M$ in A, and*

3. *$\exists f : A \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x \in A$.*

Then f is bounded, measurable, with finite support and $\lim_{n\to\infty} \int_A f_n = \int_A f$.

Remark. $\int_0^1 n\chi_{[0,\frac{1}{n}]}(x)dx = 1$ but $n\chi_{[0,\frac{1}{n}]}(x) \xrightarrow{n\to\infty} 0 \forall x \in (0, 1]$ i.e. for a.e. $x \in [0, 1]$.

Theorem 3.6. *If $A = [a, b], a < b \in \mathbb{R}$ then every bounded function $f : A \to \mathbb{R}$ that is Riemann integrable is measurable and its Riemann $\int_A f$ is equal to its Lebesgue integral $\int_A f$.*

Case of a nonnegative measurable function

Definition 3.4. Let $f : A \to [0, \infty]$ be measurable. Then we define the *integral of f over A* as

$$\int_A f = \sup \left\{ \int_A h : h : A \to [0, \infty) \text{ bounded, measurable, with finite support}, h \leq f \text{ on } A \right\}$$

For every $B \subseteq A$, we define $\int_B f = \int_A \tilde{f}$ where $\tilde{f}(x) = \begin{cases} f(x) & x \in B \\ 0 & x \notin B \end{cases}$ ($\tilde{f} = \chi_B f$ if $f < \infty$). We say that f is *integral over B* if $\int_B f < \infty$.

Proposition 3.7. *Let $f, g : A \to [0, \infty]$ be measurable. Then*

1. *$\forall \alpha, \beta \geq 0$, $\alpha f + \beta g$ is nonnegative measurable and $\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$.*
2. *$\forall B_1, B_2 \subseteq A$ measurable disjoint, $\int_{B_1 \cup B_2} f = \int_{B_1} f + \int_{B_2} f$.*
3. *If $f \leq g$ on A then $\int_A f \leq \int_A g$. Moreover, if $f = g$ a.e. on A then $\int_A f = \int_A g$. In particular, if $f = 0$ a.e. on A then $\int_A f = 0$.*

Remark. For every $A \subseteq \mathbb{R}^d$ measurable, $\int_{\mathbb{R}^d} \chi_A = m(A)$.

Theorem 3.8 (Chebyshev's Inequality)**.** *Let $f : A \to [0, \infty]$ be measurable. Then*

$$\forall c > 0 : m(f^{-1}([0, +\infty])) \leq \frac{1}{c} \int_A f$$

Corollary 3.9. *Let $f : A \to [0, \infty]$ be measurable. Then $\int_A f = 0 \Leftrightarrow f = 0$ a.e. in A.*

Corollary 3.10. *Let $f : A \to [0, \infty]$ be measurable. If f is integrable then $f < \infty$ a.e. in A.*

Theorem 3.11 (Fatou's Lemma)**.** *Let $(f_n)_{n \in \mathbb{N}}$ be measurable nonnegative on $A \subseteq \mathbb{R}^d$. Then*

$$\int_A \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int_A f_n$$

Remark. There is not equality since

$$\int_{\mathbb{R}} n\chi_{(0, \frac{1}{n})} = 1 > \int_{\mathbb{R}} \lim_{n \to} (n\chi_{(0, \frac{1}{n})}) = \int_{\mathbb{R}} 0 = 0$$

Theorem 3.12 (Monotone Convergence Theorem)**.** *Let* $(f_n)_{n\in\mathbb{N}}$ *be measurable, nonnegative functions increasing in* n *(i.e.* $f_{n+1} \geq f_n$ *on* A*). Then* $\lim_{n\to\infty} \int_A f_n = \int_A \lim_{n\to\infty} f_n$.

Corollary 3.13. *Let* $(u_n)_{n\in\mathbb{N}}$ *be measurable, nonnegative function. Then* $\sum_{n=1}^{\infty} \int_A u_n = \int_A \sum_{n=1}^{\infty} u_n$.

Case of a sign-changing function

Definition 3.5. Let $f : A \to \overline{\mathbb{R}}$ be measurable. We say that f is *integrable* if $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$ are integrable. We then call *integral* of f over A the number $\int_A f = \int_A f_+ - \int_A f_-$. For every $B \subseteq A$, we denote $\int_B f = \int_B f_+ - \int_B f_-$.

Proposition 3.14. f *integrable* $\Leftrightarrow |f|$ *integrable.*

Remark. If $f, g : A \to \overline{\mathbb{R}}$ then

$$\begin{cases} f+g \text{ is not defined on } N = \{x \in A : f(x) = -g(x) = \pm\infty\} \\ fg \text{ is not defined on } N = \{x \in A : |f(x)| = \infty, g(x) = 0 \vee |g(x)| = \infty, f(x) = 0\} \end{cases}$$

However, if f, g integrable then $|f| < \infty$ and $|g| < \infty$ a.e. in A, in which case we say $f + g, fg$ integrable and we denote $\int_A (f + g) = \int_{A\setminus N} (f + g)$ and $\int_A fg = \int_{A\setminus N} fg$.

Proposition 3.15. *Let* $f, g : A \to \overline{\mathbb{R}}$ *be integrable. Then*

1. $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ *is integrable and* $\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$.
2. $\forall B_1, B_2 \subseteq A$ *measurable disjoint,* $\int_{B_1\cup B_2} f = \int_{B_1} f + \int_{B_2} f$.
3. $f \leq g$ *on* $A \Rightarrow \int_A f \leq \int_A g$. $f = g$ *a.e. on* $A \Rightarrow \int_A f = \int_A g$.
4. $|\int_A f| \leq \int_A |f|$.

Theorem 3.16 (Dominated Convergence Theorem)**.** *Let* $(f_n)_{n\in\mathbb{N}}$ *be measurable functions on* A *such that*

1. $\exists f : A \to \overline{\mathbb{R}}$ *measurable such that* $\lim_{n\to\infty} f_n(x) = f(x)$ *for a.e.* $x \in A$, *and*
2. $\exists g : A \to \overline{\mathbb{R}}$ *integrable such that* $|f_n(x)| \leq g(x)$ *for a.e.* $x \in A$ *and* $\forall n \in \mathbb{N}$.

Then f_n *and* f *are integrable and* $\lim_{n\to\infty} \int_A f_n = \int_A f$.

Corollary 3.17 (continuity of the integral). *Let f be integrable over $A \subseteq \mathbb{R}^d$. Then*

1. *If $(A_n)_{n\in\mathbb{N}}$ is a sequence of measurable subsets of A such that $A_n \subseteq A_{n+1}$ then*

$$\int_{\bigcup_{n=1}^{\infty} A_n} f = \lim_{n\to\infty} \int_{A_n} f$$

2. *If $(A_n)_{n\in\mathbb{N}}$ is a sequence of measurable subsets of A such that $A_n \supseteq A_{n+1}$ then*

$$\int_{\bigcap_{n=1}^{\infty}} A_n f = \lim_{n\to\infty} \int_{A_n} f$$

4 Fubini and Tonelli's Theorems

Definition 4.1. Let $d_1, d_2 \in \mathbb{N}$ be such that $d = d_1 + d_2$. We denote $(x, y) \in \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. For every $E \subseteq \mathbb{R}^d$, we denote $E_x = \{y \in \mathbb{R}^{d_2} : (x, y) \in E\}$ and $E_y = \{x \in \mathbb{R}^{d_1} : (x, y) \in E\}$. $\forall f : E \to \overline{\mathbb{R}}$, $f_x : E_x \to \overline{\mathbb{R}}, y \mapsto f(x, y)$ and $f_y : E_y \to \overline{\mathbb{R}}, x \mapsto f(x, y)$

Remark. E_x and E_y are not necessarily measurable when E is measurable.

Remark. It is not always true that $\int_A(\int_B f(x, y)dy)dx = \int_B(\int_A f(x, y)dx)dy$ even when the integrals are well-defined.

Theorem 4.1 (Fubini). *Let $f : \mathbb{R}^d \to \overline{R}$ be integrable. Then*

1. *For a.e. $y \in \mathbb{R}^{d_2}$, f_y is integrable on $\mathbb{R}^{d_1}$,*

2. *$y \mapsto \int_{\mathbb{R}^{d_1}} f_y = \int_{\mathbb{R}^{d_1}} f(x, y)dx$ is integrable on $\mathbb{R}^{d_2}$, and*

3. *$\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}} f(x, y)dx)dy = \int_{\mathbb{R}^d} f$.*

Remark. The roles of x and y can be interchanged so that $\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_1}}(\int_{\mathbb{R}^{d_2}} f(x, y)dy)dx$.

Theorem 4.2 (Tonelli). *Let f be nonnegative measurable on $\mathbb{R}^d$. Then*

1. *For a.e. $y \in \mathbb{R}^{d_2}$, f_y is measurable in $\mathbb{R}^{d_1}$,*

2. *$y \mapsto \int_{\mathbb{R}^{d_1}} f_y$ is measurable in $\mathbb{R}^{d_2}$, and*

3. *$\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}} f_y) = \int_{\mathbb{R}^d} f$.*

Corollary 4.3. *If $A \subseteq \mathbb{R}^d$ is measurable then for a.e. $y \in \mathbb{R}^{d_2}$, A_y is measurable and moreover, $y \mapsto m(A_y)$ is measurable and $m(A) = \int_{\mathbb{R}^{d_2}} m(A_y)dy$.*

Corollary 4.4 (Tonelli for $A \subseteq \mathbb{R}^d$)**.** *Let $f : A \to \overline{\mathbb{R}}$ be nonnegative measurable. Then*

1. *For a.e. $y \in \mathbb{R}^{d_2}$, f_y is measurable in $\mathbb{R}^{d_1}$,*
2. *$y \mapsto \int_{\mathbb{R}^{d_1}} f_y$ is measurable in $\mathbb{R}^{d_2}$, and*
3. *$\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}} f_y) = \int_{\mathbb{R}^d} f$.*

Corollary 4.5 (Fubini for $A \subseteq \mathbb{R}^d$)**.** *Let $f : A \to \overline{\mathbb{R}}$ be integrable over A. Then*

1. *For a.e. $y \in \mathbb{R}^{d_2}$, f_y is integrable on $\mathbb{R}^{d_1}$,*
2. *$y \mapsto \int_{\mathbb{R}^{d_1}} f_y = \int_{\mathbb{R}^{d_1}} f(x,y)dx$ is integrable on $\mathbb{R}^{d_2}$, and*
3. *$\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}} f(x,y)dx)dy = \int_{\mathbb{R}^d} f$.*

Lemma 4.6. $\forall E_1 \subseteq \mathbb{R}^{d_1}, E_2 \subseteq \mathbb{R}^{d_2}$,

$$m_*(E_1 \times E_2) \leq \begin{cases} m_*(E_1)m_*(E_2) & m_*(E_1) \neq 0 \wedge m_*(E_2) \neq 0 \\ 0 & \textit{otherwise} \end{cases}$$

Theorem 4.7. *Let $E_1 \subseteq \mathbb{R}^{d_1}$ and $E_2 \subseteq \mathbb{R}^{d_2}$ be measurable. Then $E_1 \times E_2$ is measurable and*

$$m_*(E_1 \times E_2) = \begin{cases} m_*(E_1)m_*(E_2) & m_*(E_1) \neq 0 \wedge m_*(E_2) \neq 0 \\ 0 & \textit{otherwise} \end{cases}$$

Corollary 4.8. *Let $E_1 \subseteq \mathbb{R}^{d_1}$ and $E_2 \subseteq \mathbb{R}^{d_2}$ be measurable and f be a measurable function on E_1. Then $\tilde{f} : E_1 \times E_2 \to \overline{\mathbb{R}}, (x,y) \mapsto f(x)$ is measurable on $E_1 \times E_2$.*

Proposition 4.9. *Let $d_1 = d - 1$, $A \subseteq \mathbb{R}^{d_1}$ be measurable and $f : A \to [0,\infty]$. Then f is measurable if and only if $E = \{(x,y) \in A \times \mathbb{R} : 0 \leq y \leq f(x)\}$ is measurable. Furthermore, if f is measurable, then $m(E) = \int_A f(x)dx$.*

Proposition 4.10. *Let f be measurable on $\mathbb{R}^d$. Then $g : \mathbb{R}^{2d} \to \overline{\mathbb{R}}, (x,y) \mapsto f(x-y)$ is measurable.*

Remark. This is useful when defining convolution $f * g : x \mapsto \int_{\mathbb{R}^d} f(x-y)g(y)dy$.

5 Differentiation

Theorem 5.1. *A monotone function $f : [a, b] \to \mathbb{R}$ is differentiable almost everywhere in (a, b). Furthermore, f' is integrable and*

$$\int_a^b f' \begin{cases} \leq f(b) - f(a) & f \text{ increasing} \\ \geq f(b) - f(a) & f \text{ decreasing} \end{cases}$$

Remark. The Cantor-Lebesgue function is monotone, differentiable in $[0, 1]$, with $\varphi' = 0$ a.e. in $[0, 1]$ but $\int_0^1 \varphi' = 0 < \varphi(1) - \varphi(0) = 1$.

Theorem 5.2. *Let F be a collection of bounded intervals in $[a, b] \subseteq \mathbb{R}$ of positive length. Then there exists a countable collection $F' \subseteq F$ of disjoint intervals such that $\bigcup_{I \in F} I \subseteq \bigcup_{I \in F'} 5I$, where $5I = \{x \in \mathbb{R} : x_I + \frac{1}{5}(x - x_I) \in I\}$ (x_I middle point of I).*

Remark. It is possible to replace 5 by a number $x > 3$ but no less: consider $F = \{[-1, 0], [0, 1]\}$.

Proposition 5.3. *A monotone function $f : [a, b] \to \mathbb{R}$ has at most countably many discontinuities.*

Functions of bounded variation

Definition 5.1. Let $f : [a, b] \to \mathbb{R}$ be a function. We call *total variation* of f on $[a, b]$ the number

$$T_f(a, b) = \sup \left\{ \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| : a = x_0 < x_1 < \cdots < x_k = b \right\}$$

If $T_f(a, b) < \infty$, then we say that f is of *bounded variation* on $[a, b]$.

Remark. Monotone and Lipschitz continuous functions are of bounded variation.

Remark.

$$f(x) = \begin{cases} x \cos(\frac{1}{x}) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

is not of bounded variation.

Theorem 5.4. *A function $f : [a,b] \to \mathbb{R}$ is of bounded variation if and only if it can be written as the difference between two increasing functions. In particular, if f is of bounded variation then f is differentiable a.e. and f' is integrable over $[a,b]$.*

Absolutely continuous functions

Definition 5.2. We say that a function $f : [a,b] \to \mathbb{R}$ is *absolutely continuous* on $[a,b]$ if $\forall \epsilon > 0 \exists \delta > 0$ such that for every finite collection of disjoint open bounded intervals $(a_k, b_k) \subseteq [a,b], 1 \leq k \leq n$, if $\sum_{k=1}^{n}(b_k - a_k) < \delta$ then $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$.

Remark. f absolutely continuous $\Rightarrow$ f uniformly continuous by taking $n = 1$.

Remark. The Cantor-Lebesgue function φ is not absolutely continuous on $[0,1]$.

Proposition 5.5. *If $f : [a,b] \to \mathbb{R}$ is Lipschitz continuous then f is absolutely continuous on $[a,b]$.*

Theorem 5.6. *If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ then f can be written as the difference between two increasing absolutely continuous functions. In particular, f is of bounded variation on $[a,b]$.*

Theorem 5.7. *Let $f : [a,b] \to \mathbb{R}$.*

1. *If f is absolutely continuous on $[a,b]$ then*

$$\forall x \in [a,b] : \int_{[a,x]} f' = f(x) - f(a)$$

2. *Conversely, for every integrable function g over $[a,b]$, the function $x \mapsto \int_a^x g$ is absolutely continuous on $[a,b]$ with derivative equal to g a.e. in $[a,b]$.*

Lemma 5.8. *Let h be integrable over $[a,b]$. Then $h = 0$ a.e. in $[a,b] \Leftrightarrow \int_a^x h = 0$ for all $x \in (a,b)$.*

Corollary 5.9. *If $f : [a,b] \to \mathbb{R}$ is monotone, then f is absolutely continuous in $[a,b] \Leftrightarrow \int_a^b f' = f(b) - f(a)$.*

Corollary 5.10 (Lebesgue decomposition)**.** *Every function $f : [a,b] \to \mathbb{R}$ of bounded variations can be written as $f = f_{abs} + f_{sing}$, where $f_{abs} = \int_a^x f'$ is absolutely continuous in $[a,b]$ and $f_{sing} = f - f_{abs}$ is such that $f'_{sing} = 0$ a.e. in $[a,b]$.*