Honours Analysis 3

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December 7, 2021

1 Borel Sets

We will work for some time on $\mathbb{R}$ exclusively. Before beginning Measure Theory: a quick recap of Topology.

Definition 1.1 (Open Set)**.** *A subset $U \subset \mathbb{R}$ is called* open *if either $U = \emptyset$ or else*

$$\forall x \in U, \exists r > 0 \text{ such that } (x - r, x + r) \subset U$$

Some examples of open sets: $\emptyset, \mathbb{R}, (a, b), (a, \infty), (-\infty, a)$. There are many more because any union of an open set is still open and any finite intersection of open sets is open.

Definition 1.2 (Closed Set)**.** *$F \subset \mathbb{R}$ is called* closed *if $\mathbb{R} \setminus F := F^c$ is open.*

F is closed $\iff$ F contains all points $x \in \mathbb{R}$ which have the property that $\forall r > 0, (x - r, x + r) \cap F \neq \emptyset$.

If $F \subset \mathbb{R}$ is any set, the closure of F, denoted by $\overline{F}$, is the smallest closed set that contains F.

Definition 1.3 (Compact)**.** *A subset $G \subset \mathbb{R}$ is* compact *if given any collection $\{U_i\}_{i \in I}$ of open sets $U_i \subset \mathbb{R}$ with $G \subset \cup_{i \in I} U_i$, there exists $J \subset I$, J finite, such that $G \subset \cup_{j \in J} U_j$*

*Notes from the lectures of Valentino Tosatti

Theorem 1.1 (Heine-Borel)**.** *$G \subset \mathbb{R}$ is compact $\iff$ G is closed and bounded. To be bounded means $G \subset (a,b)$ for some $a, b \in \mathbb{R}$.*

Corollary 1.1.1 (Nested Set Theorem)**.** *Let $\{F_n\}_{n=1}^{\infty}$ be a countable collection of non-empty, bounded, closed sets $F_n \subset \mathbb{R}$ with $F_{n+1} \subset F_n \forall n$, then*

$$\cap_{n=1}^{\infty} F_n \neq \emptyset$$

Proof. Suppose $\cap_{n=1}^{\infty} F_n = \emptyset$ so let $U_n = F_n^c$ be open sets, such that $\cup_{n=1}^{\infty} U_n = \mathbb{R}$. We also have that $U_n \subset U_{n+1}$, since the F_n were nested. Now F_1 is compact by Heine-Borel and $F_1 \subset \cup_{n=1}^{\infty} U_n \Rightarrow$ by compactness I can find a finite subcover of F_1, say $F \subset \cup_{n=1}^{N} U_n = U_N = F_N^c$

On the other hand $F_N \subset F_1$ by the nested property which implies $F_N = \emptyset$ which is a contradiction. □

2 Measure Theory

We want to measure the size of a set. We will deal with a subset of $\mathbb{R}$.

It turns out that one needs to select a class of subsets of $\mathbb{R}$ that one wants to measure. This class of subsets will have certain properties which are as follows.

Definition 2.1 (σ-algebra)**.** *A collection $\mathcal{A}$ of subsets of $\mathbb{R}$ is called a σ-algebra if it satisfies*

1. *$\emptyset \in \mathcal{A}$*
2. *If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$*
3. *If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ then $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$*

Observe the following:

- $\mathbb{R} \in \mathcal{A}$ always

- If $\{A_n\}_{n=1}^N \subset \mathcal{A}$ then $\cup_{n=1}^N A_n \in \mathcal{A}$ (just define $A_n = \emptyset$ for $n > N$)
- If $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$ then $\cap_{n=1}^\infty A_n \in \mathcal{A}$ (since $(\cap_{n=1}^\infty A_n)^c = \cup_{n=1}^\infty A_n^c$)
- If $A, B \in \mathcal{A}$ then $A \setminus B \in \mathcal{A}$ too since $A \setminus B = A \cap B^c$

Examples:

1. $\mathcal{A} = \{\emptyset, \mathbb{R}\}$ "Minimal σ-algebra"
2. $\mathcal{A} = \mathcal{P}(\mathbb{R})$ = Collection of all subsets of $\mathbb{R}$. "Maximum σ-algebra"

In fact, if $\mathcal{A}$ is any σ-algebra, then $\{\emptyset, \mathbb{R}\} \subseteq \mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$

For better examples, let F be any collection of subsets of $\mathbb{R}$. I want to make F into a σ-algebra. Define $m = \{\mathcal{A} \mid \mathcal{A} \text{ is a } \sigma\text{-algebra that satisfies } F \subset \mathcal{A}\}$. $m \neq \emptyset$ since it contains $\mathcal{P}(\mathbb{R})$

If $\mathcal{A}, \mathcal{B} \in m$, I can define $\mathcal{A} \cap \mathcal{B} = \{A \subset \mathbb{R} \mid A \in \mathcal{A} \text{ and } A \in \mathcal{B}\}$ and I can do the same for $\cap_{i \in I} \mathcal{A}$ arbitrary intersection of σ-algebra is still a σ-algebra

Define $\hat{F}_i = \cap_{\mathcal{A} \in m} \mathcal{A}$ as a σ-algebra and $F \subset \hat{F}$ and it is the minimal σ-algebra with these properties. If G is a σ-algebra with $F \subset G$, then $\hat{F} \subset G$. $\hat{F}$ is the σ-algebra generated by F. Concretely, $\hat{F}$ consists of all subsets of $\mathbb{R}$ that can be constructed by applying countable unions, intersections, and complements to elements of F.

Definition 2.2 (Borel Sets)**.** *The σ-algebra $\mathcal{B}$ of Borel Sets is the σ-algebra $\hat{F}$ generated by*

$$F = \{U \subset \mathbb{R} \mid U \text{ open }\}$$

Remark. *$\mathcal{B}$ is also the σ-algebra generated by the family of all closed subsets of $\mathbb{R}$*

Singletons $\{x\} \subset \mathbb{R}$ are closed so if $A \subset \mathbb{R}$ is at most countable then A is Borel. (e.g $\mathbb{Q} \subset \mathbb{R}$) (e.g $\mathbb{R} \setminus \mathbb{Q}$)

Not all Subsets of $\mathbb{R}$ are Borel. One can actually show that the cardinality of $\mathcal{B}$ is the same as the cardinality of $\mathbb{R}$. On the other hand $\mathcal{P}(\mathbb{R})$ has strictly larger cardinality.

3 Lebesgue Outer Measure

We are hoping to measure the size of subsets of $\mathbb{R}$. Ideally we would like to find or construct a function

$$m : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_{\geq 0} \cup \{+\infty\} = [0, \infty]$$

Which satisfies the following measure requirements:

1. If $I = [a, b]$ or (a, b) or $[a, b)$, or $(a, b]$, $a, b \in \mathbb{R}, a \leq b$ then $m(I) = b - a =$ measure of interval
2. m is translation invariant. i.e if $E \subset \mathbb{R}$ and $x \in \mathbb{R}$, let $E + x = \{y + x \mid y \in E\}$ then $m(E + x) =$ m(E)
3. If $\{E_j\}_{j=1}^n$ is a finite collection of pairwise disjoint $E_j \subset \mathbb{R}$ then

$$m\left(\cup_{j=1}^n E_j\right) = \sum_{j=1}^n m(E_j)$$

4. The same as (3) except for $n = \infty$

Theorem 3.1. *There is no such m satisfying all 4 requirements*

The proof for this will come later. The solution for this is that we do not try to measure all subsets of $\mathbb{R}$. So we have $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ but now we will just be happy with $m : \mathcal{A} \to [0, \infty]$ where $\mathcal{A}$ is a σ-algebra which has enough elements. For example $\mathcal{A} > \mathcal{B}$.

We will follow H. Lebesgue as we proceed in two steps.

Step 1: construct Lebesgue outer measure $m^\star : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ satisfying requirements 1,2, and 3.

Step 2: Use $m^\star$ to define $\mathcal{A}$ and let $m \subset m^\star \mid \mathcal{A}$

To create this Lebesgue outer measure on $\mathbb{R}$ we satisfy a weakened version of requirement (3) that can be called (3w). For any countably infinite collection $\{E_j\}_{j=1}^\infty$ of arbitrary subsets $E_j \subset \mathbb{R}$

$$m^\star(\cup_{j=1}^\infty E_j) \leq \sum_{j=1}^\infty m(E_j)$$

Theorem 3.2 (Lebesgue Outer Measure)**.** *There is a map* $m^\star : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ *that satisfies the measure requirements 1, 2, and 3w.*

This $m^\star$ is called the Lebesgue outer measure on $\mathbb{R}$.

How do we define outer measure $m^\star(A)$?

Observe that any $A \subseteq \mathbb{R}$ can be covered by some countable infinite collection $\{I_j\}_{j=1}^\infty$ of bounded open intervals, which are allowed to be empty, but we do not assume that I_j be pairwise disjoint.

For example: $I_j = (-j, j),\ j = 1, 2, 3 \dots$

Let

$$\mathcal{C}_A = \{\{I_j\}_{j=1}^\infty \mid I_j \text{ bounded open intervals such that } A \subset \cup_{j=1}^\infty I_j\}$$

$\mathcal{C}_A \neq \emptyset$ by our example so for each $\{I_j\} \in \mathcal{C}_A$, I can consider

$$\sum_{j=1}^\infty \ell(I_j) \in \mathbb{R}_{\geq 0} \cup \{+\infty\} \qquad (\ell \text{ denotes length})$$

Definition 3.1 (Outer Measure)**.**

$$\boxed{m^\star(A) := \inf_{\{I_j\} \in \mathcal{C}_\mathcal{A}} \sum_{j=1}^\infty \ell(I_j)} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

This defines a map $m^\star : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$

Simple Properties:

- *Monotonicity*: If $A \subseteq B$ then $m^\star(A) \leq m^\star(B)$. Indeed by definition $\mathcal{C}_B \subseteq \mathcal{C}_A$ hence the infimum over $\mathcal{C}_B$ is $\geq$ than the infimum over $\mathcal{C}_A$.
- *Empty Set*: $m^\star(\emptyset) = 0$. Given any $1 > \epsilon > 0$, let $I_j = (-\epsilon^j, \epsilon^j),\ j = 1, 2, \dots$ $\{I_j\} \in \mathcal{C}_\emptyset$ and $\sum_{j=1}^\infty \ell(I_j) = 2\sum_{j=1}^\infty \epsilon^j = \frac{2\epsilon}{1-\epsilon}$ from the geometric series going to zero so $m^\star(\emptyset) \leq \frac{2\epsilon}{1-\epsilon}$ $\forall 0 < \epsilon < 1$

- If $A \in \mathbb{R}$ is finite or countable infinite then $m^\star(A) = 0$. Indeed enumerate all elements of A by $\{a_j\}_{j=1}^\infty$. (If A is finite say $|A| = n$ let $a_j = a_n$ for all $j > n$). For any $0 < \epsilon < 1$, let $I_j = \left(-\epsilon^j + a_j, a_j + \epsilon^j\right)$ so $A \subseteq \cup_{j=1}^\infty I_j$ and $\sum_{j=1}^\infty \ell(I_j) = \frac{2\epsilon}{1-\epsilon}$ hence as before, $m^\star(A) = 0$. For example $m^\star(\mathbb{Q}) = 0$

We will now prove that the Lebesgue outer measure satisfies 1, 2, and 3w of the measure requirements.

Proof of Property 1: i.e $m^\star(I) = \ell(I)$ for any interval $I \subseteq \mathbb{R}$

Assume that $I = [a, b]$, $a < b$ are finite numbers. Assume that I is a bounded closed interval. Our goal is to show that $m^\star(I) = b - a$. One direction of inequality is easy to prove, the other is quite tedious and will be left out.

For any $\epsilon > 0$ let $I_1 = (a - \epsilon, b + \epsilon) > I$, let $I_j = \emptyset, j \geq 2$ so $\{I_j\} \in \mathcal{C}_I \Rightarrow m^\star(I) \leq \sum_{j=1}^\infty \ell(I_j) = b - a + 2\epsilon$. Let $\epsilon \to 0$ and we obtain $m^\star(I) \leq b - a$.

Proof of Property 2: i.e $\forall A \subset \mathbb{R}, \forall x \in \mathbb{R}$, $m^\star(A + x) = m^\star(A)$

$\mathcal{C}_A$ and $\mathcal{C}_{A+x}$ are naturally in bijection via $\{I_j\} \leftrightarrow \{I_j + x\}$. Furthermore $\ell(I_j + x) = \ell(I_j)$

$$\begin{aligned} m^\star(A + x) &= \inf_{\{I_j + x\} \in \mathcal{C}_{A+x}} \sum_{j=1}^\infty \ell(I_j + x) \\ &= \inf_{\{I_j\} \in \mathcal{C}_A} \sum_{j=1}^\infty \ell(I_j) = m^\star(A) \end{aligned}$$

Proof of Property 3w: i.e If $\{E_j\}_{j=1}^n$ is a finite collection of pairwise disjoint $E_j \subset \mathbb{R}$ then $m^\star\left(\cup_{j=1}^n E_j\right) \leq \sum_{j=1}^n m^\star(E_j)$

If $m^\star(E_j) = +\infty$ for some j, then the property holds. We may assume that $m^\star(E_j) < +\infty \; \forall j$. Let $\epsilon > 0$. By the definition of infimum, for each $j \geq 0$, there is

$$\{I_{j,k}\}_{k=1}^\infty \in \mathcal{C}_{E_j} \text{ such that } \sum_{k=1}^\infty \ell(I_{j,k}) < m^\star(E_j) + \epsilon 2^{-j}$$

Thus $\{I_{j,k}\}_{k=1}^{\infty}$ is still countable and it covers $\cup_{j=1}^{\infty} E_j$ meaning it belongs to $\mathcal{C}_{\cup_{j=1}^{\infty}} E_j$, so by definition

$$m^{\star}\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \ell(I_{j,k}) < \sum_{j=1}^{\infty}(m^{\star}(E_j) + \epsilon 2^{-j}) = \sum_{j=1}^{\infty} m^{\star}(E_j) + \epsilon$$

Then let $\epsilon \to 0$. Clearly, by taking all $E_j = \emptyset$ except finitely many, we have the same subadditivity 3w for finite collections.

Corollary 3.2.1. $m^{\star}([0,1] \cap (\mathbb{R} \setminus \mathbb{Q})) = 1 = \ell([0,1])$

Proof.

$$\begin{aligned} m^{\star}([0,1] \cap (\mathbb{R} \setminus \mathbb{Q})) &\leq m^{\star}([0,1]) = 1 \\ &\leq m^{\star}([0,1] \cap (\mathbb{Q})) + m^{\star}([0,1] \cap (\mathbb{R} \setminus \mathbb{Q})) \\ &\leq 0 + 1 \end{aligned}$$

□

Corollary 3.2.2. $\mathbb{R} \setminus \mathbb{Q}$ *is uncountable*

Proof. If not, then

$$m^{\star}(\mathbb{R} \setminus \mathbb{Q}) = 0 \geq m^{\star}([0,1] \cap (\mathbb{R} \setminus \mathbb{Q})) = 1$$ □

4 The σ-Algebra Of Lebesgue Measurable Sets

$m^{\star}$ does not satisfy the third measurability requirement without the weak 3w condition. We can construct some examples to prove this. $A, B \subset \mathbb{R}, A \cap B = \emptyset$, such that $m^{\star}(A \cup B) < m^{\star}(A) + m^{\star}(B)$ later in the class.

The idea to avoid this problem is to look at "reasonable" subsets of $\mathbb{R}$ for which this paradox disappears.

Definition 4.1 (Carathéodory)**.** *$E \subseteq R$ is called (Lebesgue) measurable if* $\forall A \subset \mathbb{R}$

$$m^{\star}(A) = m^{\star}(A \cap E) + m^{\star}(A \cap E^c)$$

Remark. *This is equivalent to Lebesgue's definition: E is measurable if and only if*

$$\exists U \subset \mathbb{R} \text{ such that } E \subset U \text{ and } m^\star (U \setminus E) < \epsilon$$

But we will discuss this later.

Suppose that A is measurable and $B \subset \mathbb{R}$ is any set such that $A \cap B = \emptyset$ then

$$m^\star (A \cup B) = m^\star \left(\underbrace{(A \cup B) \cap A}_{=A} \right) + m^\star \left(\underbrace{(A \cup B) \cap A^c}_{=B} \right)$$

Going back to our counter example for $m^\star$ and measurability requirement 3, A or B would have to be unmeasurable.

Here's another observation: For $E, A \subset \mathbb{R}$ arbitrary sets we have

$$A = (A \cap E) \cup (A \cap E^c)$$

So by 3w $m^\star (A) \leq m^\star (A \cap E) + m^\star (A \cap E^c)$, so E is measurable $\iff$ $\forall A \subset \mathbb{R}$

$$\boxed{m^\star (A) \geq m^\star (A \cap E) + m^\star (A \cap E^c)}$$

This holds trivially for $m^\star (A) = \infty$

Example 1: $\emptyset$ is measurable. $\forall A \subset \mathbb{R}$

$$m^\star (A) = \cancel{m^\star (A \cap \emptyset)} + m^\star (A \cap \mathbb{R})$$

Example 2: $\mathbb{R}$ is measurable. $\forall A \subset \mathbb{R}$

$$m^\star (A) = m^\star (A \cap \mathbb{R}) + m^\star (A \cap \emptyset)$$

Proposition. *$E \subset \mathbb{R}$ with $m^\star (E) = 0$, then E is measurable.*

Corollary. *Every countable set is measurable. $\mathbb{Q}$ measurable $\to \mathbb{R} \setminus \mathbb{Q}$ are measurable*

Proof. Let $A \subset \mathbb{R}$ be any set

$$A \cap E \subset E \Rightarrow m^\star (A \cap E) \leq m^\star (E) = 0$$
$$A \cap E^c \subset A \Rightarrow m^\star (A \cap E^c) \leq m^\star (A)$$
$$\text{So } m^\star (A) \geq m^\star (A \cap E^c) + \cancel{m^\star (A \cap E)}$$

$\square$

Our goal is to show that Lebesgue measurable sets $\mathcal{L} = \{E \subset \mathbb{R} \mid E \text{ is measurable}\}$ is a σ-algebra on $\mathbb{R}$. We just need to show that if $\{E_j\}_{j=1}^{\infty}$ with $E_j \in \mathcal{L}$, $\forall j$, then $\cup_{j=1}^{\infty} E_j \in \mathcal{L}$

Proposition. *If* $\{E_j\}_{j=1}^{n} \subset \mathcal{L}$ *then* $\cup_{j=1}^{n} E_i \in \mathcal{L}$

Proof. We use mathematical induction. $n = 1$ is trivial so we set the base case as $n = 2$. E_1, E_2 are measurable, Let $A \subset \mathbb{R}$ be any set

$$\begin{aligned} m^{\star}(A) &= m^{\star}(E_1 \cap A) + m^{\star}(A \cap E_1^c) \\ &= m^{\star}(A \cap E_1) + m^{\star}((A \cap E_1^c) \cap E_2) + m^{\star}((A \cap E_1^c) \cap E_2^c) \\ &= m^{\star}(A \cap E_1) + m^{\star}((A \cap E_1^c) \cap E_2) + m^{\star}(A \cap (E_1^c \cap E_2^c)) \\ &= m^{\star}(A \cap E_1) + m^{\star}((A \cap E_1^c) \cap E_2) + m^{\star}(A \cap (E_1 \cup E_2)^c) \\ &\geq m^{\star}(A \cap (E_1 \cup E_2)) + m^{\star}(A \cap (E_1 \cup E_2)^c) \end{aligned} \tag{3w}$$

So $E_1 \cup E_2 \in \mathcal{L}$.

Induction step $n \geq 2$

$$\bigcup_{j=1}^{n} E_j = \left(\bigcup_{j=1}^{n-1} E_j\right) \cup E_n \in \mathcal{L} \text{ by the } n = 2 \text{ case}$$ □

To prove that this also applies to countable sets, we use

Proposition (Analog of measurability requirement 3 for $m^{\star} \mid \mathcal{L}$)**.** *Suppose* $A \subset \mathbb{R}$ *is any set and* $\{E_j\}_{j=1}^{n}$ *is a finite disjoint collection of sets* $E_j \in \mathcal{L}$, *then*

$$m^{\star}\left(A \cap \bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m^{\star}(A \cap E_j)$$

In particular take $A = \mathbb{R}$ *to get* $m^{\star}\left(\bigcup_{j=1}^{n} E_j\right) = \sum m^{\star}(E_j)$

Proposition. *If* $\{E_j\}_{j=1}^{\infty}$ *is a countable family with* $E_i \in \mathcal{L}\ \forall j$, *then* $\cup_{j=1}^{\infty} E_j \in \mathcal{L}$. *In particular,* $\mathcal{L}$ *is a* σ*-algebra.*

We would like to have the Borel sets be measurable, i.e $\mathcal{B} \subset \mathcal{L}$. Recall that $\mathcal{B} = \hat{\mathcal{F}}$, where $\mathcal{F} = \{U \subset \mathbb{R} \mid U \text{ is open }\}$ and ˆ denotes the σ-algebra.

This results follows from the measurability of intervals combined with the measurability of the union of measurable sets.

Proposition. *If $I \subseteq \mathbb{R}$ is any interval, then I is measurable.*

Theorem 4.1. *$\mathcal{L}$ = Lebesgue Measurable subsets of $\mathbb{R}$ form a σ-algebra that contains the Borel σ-algebra $\mathcal{B}$*

Proof. We already know that $\mathcal{L}$ is a σ-algebra. If we can show that $\mathcal{L}$ contains all open sets $U \subset \mathbb{R}$, then $\mathcal{L}$ (being a σ-algebra) must contain $\mathcal{B}$ which is the σ-algebra generated by open sets. Now if $U \subset \mathbb{R}$ is any (non empty) open set then by definition $\forall x \in U, \exists I_x \ni x$ where I_x is an open interval and $I_x \subset U$.

We want to choose I_x to be the "maximal" such. So by assigning

$$a_x \coloneqq \inf\{z \in \mathbb{R} \mid (z, x) \subset U\} \text{ satisfies } a_x < x$$

and

$$b_x \coloneqq \sup\{y \in \mathbb{R} \mid (x, y) \subset U\} \text{ satisfies } x < b_x$$

so $I_x \coloneqq (a_x, b_x)$ is an open interval that contains x and by construction $I_x \in U$. It is the largest such, in the sense that if $a_x > -\infty$ then $a_x \notin U$ and symmetrically if $b_x < \infty$ then $b_x \notin U$.

For any $y \in I_x$, we have $y < b_x$, so there is $z > y$ such that $(x, z) \subset U$ so $y \in U$. Indeed, if $a_x \in U$ then since U open, $\exists r > 0$ such that $(a_x - r, a_x + r) \subset U$ contradicting the definition of a_x.

So $U = \cup_{x \in U} I_x$. It is a huge union, however if $x, x' \in U, x \neq x'$, then either $I_x \cap I_{x'} = \emptyset$, or if not then necessarily $I_x = I_{x'}$, since $I_x \cup I_{x'}$ is then another open interval that contains x & x' and is a subset of U, so by maximality it must equal I_x & $I_{x'}$. So, throwing away all repeated I_x, we can write $U = \cup_{i \in I} I_x$ for some I where the intervals I_{x_i} are pairwise disjoint. By density of $\mathbb{Q} \subset \mathbb{R}$, each such interval contains a different rational number $r_i \in I_{x_i}$. Since $\mathbb{Q}$ is countable, I is at worst countable.

So every U open is an at most countable disjoint union of open intervals. Since such intervals belong ot $\mathcal{L}$, and $\mathcal{L}$ is a σ-algebra, it follows that every U open is in $\mathcal{L}$ as desired. $\square$

Proposition (The σ-algebra $\mathcal{L}$ is also translation invariant)**.** *If $E \subset \mathcal{L}$ and $x \in \mathbb{R}$ then $E + x \in \mathcal{L}$*

Proof. Given any $A \subset \mathbb{R}$,

$$\begin{aligned} m^\star(A) &= m^\star(A - x) \\ &= m^\star((A - x) \cap E) + m^\star((A - x) \cap E^c) \\ &= m^\star(A \cap E + x) + m^\star(A \cap (E + x)^c) \quad (m^\star \text{ translation invariant}) \end{aligned}$$

□

Remark. *If $A \in \mathcal{L}$ with $m^\star(A) < \infty$, and $B \subset \mathbb{R}$ is any set with $A \subset B$, then*

$$m^\star(B \setminus A) = m^\star(B) - m^\star(A)$$

5 Outer and Inner Approximation of Lebesgue Measurable Sets

Definition 5.1 (Gebiet-Durchshnitt)**.** *A subset $A \subset \mathbb{R}$ is called a G_δ if $A = \cap_{i=1}^{\infty} A_i$ where A_i are all open (possibly empty).*

Definition 5.2 (Fermé-Somme)**.** *A subset $A \subset \mathbb{R}$ is called a F_σ if $A = \cup_{i=1}^{\infty} A_i$ where A_i are all closed (possibly empty).*

Clearly, A is $G_\delta \iff A^c$ is F_δ. Also clearly, all G_δ and F_σ sets are Borel. Of course not all G_δ are open, e.g $[0,1] = \cap_{i=1}^{\infty} \left(-\frac{1}{i}, 1 + \frac{1}{i}\right)$ and not all F_σ are closed. e.g. $(0,1) = \cup_{i=1}^{\infty} \left[\frac{1}{i}, 1 - \frac{1}{i}\right]$

$\mathbb{Q}$ is clearly F_σ, so $\mathbb{R} \setminus \mathbb{Q}$ is G_δ. With this, we can give several equivalent formulations of measurability.

Theorem 5.1. *Let $E \subset \mathbb{R}$ be any set, then the following are equivalent:*

1. *$E \in \mathcal{L}$*
2. *$\forall \epsilon > 0$, $\exists U \supset E$, U open, $m^\star(U \setminus E) < \epsilon$*

3. $\exists G \subset \mathbb{R}$ *a* G_δ *set,* $G \supset E$*, with* $m^\star(G \setminus E) = 0$

4. $\forall \epsilon > 0$*,* $\exists F \subset E$*,* F *closed,* $m^\star(E \setminus F) < \epsilon$

5. $\exists F \subset \mathbb{R}$ *a* F_σ *set,* $F \subset E$ *with* $m^\star(E \setminus F) = 0$

Proposition. *For an* $E \in \mathcal{L}$ *with* $m^\star(E) < \infty$*. Then* $\forall \epsilon > 0$*,* $\exists \{I_j\}_{j=1}^n$ *a finite disjoint family of open intervals so that if we let* $U = \cup_{j=1}^n I_j$ *(open) then* $m^\star(E \Delta U) < \epsilon$*.*

6 Lebesgue Measure

We can now take $m^\star$ and restrict it to $\mathcal{L}$. $m^\star|_{\mathcal{L}}$.

Definition 6.1 (Lebesgue Measure)**.** *This Lebesgue Measure is a function*

$$m := m^\star|_{\mathcal{L}} \colon \mathcal{L} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

This means that for $E \in \mathcal{L}$ we define $m(E) = m^\star(E)$. Clearly, m satisfies the measurability requirements 1, 2, & 3 as we have proved earlier. It also satisfies requirement 4 which was requirement 3 for countably infinite sets.

Proposition. *If* $\{E_j\}_{j=1}^\infty$ *is a countably infinite collection of pairwise disjoint sets* $E_j \in \mathcal{L}$ *(possibly empty), then* $\cup_{j=1}^\infty E_j \in \mathcal{L}$ *and*

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j)$$

Proof. We proved earlier that $\cup_{j=1}^\infty E_j \in \mathcal{L}$ and that

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m(E_j)$$

For the opposite inequality, for each n we proved earlier that

$$m\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m(E_j)$$

But $\cup_{j=1}^{n} E_j \subset \cup_{j=1}^{\infty} E_j$, hence

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) \geq m\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m(E_j) \ \ \forall n$$

Take the limit as $n \to \infty$ to get

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) \geq \sum_{j=1}^{\infty} m(E_j)$$

As desired. This argument shows that measurability requirement 3 and 3w together imply 4. □

7 Non-Measurable Sets

We saw earlier that if $E \subset \mathbb{R}$ satisfies $m^\star(E) = 0$ then $E \in \mathcal{L}$. In particular, $\forall F \subset E$, $m^\star(F) \leq m^\star(E) = 0$, so $F \in \mathcal{L}$ too. This however totally fails when $m^\star(E) > 0$.

Theorem 7.1 (Vitali)**.** *For any $E \subset \mathbb{R}$ with $m^\star(E) > 0$, there is an $F \subset E$ which is NOT measurable. The construction uses the axiom of choice (and it is really needed).*

The proof of this theorem and construction of a Vitali set are currently omitted due to length.

8 Cantor Set

We showed earlier that if $A \subset \mathbb{R}$ is countable then $A \in \mathcal{L}$ and $m(A) = 0$. How about the converse; if $A \in \mathcal{L}$ has $m(A) = 0$, is A countable? No!

Theorem 8.1 (Cantor)**.** *There is a closed, uncountable set $\mathcal{C}$ with $m(\mathcal{C}) = 0$*

Start with an interval $I = [0,1]$ and remove the middle $\frac{1}{3}$, namely $(\frac{1}{3}, \frac{2}{3})$.

$$\begin{aligned}
\mathcal{C}_1 &:= I \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right] \\
\mathcal{C}_2 &:= \mathcal{C}_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9}\right) \bigcup \left(\frac{7}{9}, \frac{8}{9}\right)\right) \\
\mathcal{C}_k &:= \mathcal{C}_{k-1} \setminus \bigcup_{j=0}^{3^{k-1}-1} \left(\frac{3j+1}{3^k}, \frac{3j+2}{3^k}\right) \\
&= [0,1] \setminus \bigcup_{l=1}^{k} \bigcup_{j=0}^{3^{l-1}-1} \left(\frac{3j+1}{3^l}, \frac{3j+2}{3^l}\right)
\end{aligned}$$

Thus $\{\mathcal{C}_k\}_{k=1}^{\infty}$ is a very large descending (i.e nested $\mathcal{C} \subset \mathcal{C}_{k-1}$) sequence of closed sets, and $\mathcal{C}_k$ is a disjoint union of 2^k closed intervals of length $\frac{1}{3^k}$. Let then $\mathcal{C} = \bigcap_{k=1}^{\infty} \mathcal{C}_k$, so $\mathcal{C}$ is closed, and hence also measurable.

Since $m(\mathcal{C}_k) = \left(\frac{2}{3}\right)^k$, $m(\mathcal{C}) \leq m(\mathcal{C}_k) \leq \left(\frac{2}{3}\right)^k$ $\forall k$. Taking the limit as $k \to \infty$ we get $m(\mathcal{C}) = 0$.

Suppose that $\mathcal{C}$ was countable, let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of all it's elements. Then writing $\mathcal{C}_1 =$ the disjoint union of 2 interavals, we must have that c_1 belongs to precisely one of them. Say $c_1 \notin F_1$. Now $F_1 \subset \mathcal{C}_2$ is made of 2 disjoint intervals, and one of them does not contain c_2, say $c_2 \notin F_2$.

Continue this way until we get a sequence of $\{F_k\}_{k=1}^{\infty}$, where F_k is a closed interval, $F_{k+1} \subset F_k$, and $F_k \subset \mathcal{C}_k$, and $c_k \notin F_k$. By the nested set theorem, let $x \in \cap_{k=1}^{\infty} F_k$. Then

$$x \in \cap_{k=1}^{\infty} F_k \subset \cap_{k=1}^{\infty} \mathcal{C}_k = \mathcal{C}$$

So $x \in \mathcal{C}$ but $\{c_k\}_{k=1}^{\infty}$ enumerates ALL points of $\mathcal{C}$ so $\exists n$ such that $x = c_n$. Hence $x \notin F_n$ but this is a contradiction so we conclude that $\mathcal{C}$ is uncountable.

Finally observe that $\mathcal{C}$ is closed and $\mathcal{C} \subset [0,1]$, so $\mathcal{C}$ is compact by Heine-Borel.

There are two variations of this theorem.

1. If instead of removing the middle third, we removed the middle $p\%$ where $0 < p < 100$, then we also get a Cantor set which has the same properties as $\mathcal{C}$.

2. We could also remove a *smaller* proportion at each step, instead of a fixed one. At each step we remove 2^{n-1} intervals of length a^n for some $0 < a \leq \frac{1}{3}$. Then the total length removed is $\sum_{n=1}^{\infty} 2^{n-1}a^n = \frac{a}{1-2a}$. So, for this "fat" Cantor set $m(\mathcal{C}_{\text{fat}}) = 1 - \frac{a}{1-2a} = \frac{1-3a}{1-2a}$. Which is indeed 0 when $a = \frac{1}{3}$ (standard Cantor), and $m(\mathcal{C}_{\text{fat}}) > 0$ for $0 < a < \frac{1}{3}$

Remark. $|\mathcal{L}| = |\mathcal{P}(\mathbb{R})|$*:* $\leq$ *is trivial so* $\forall A \subset \mathcal{C}$*,* $A \in \mathcal{L}$ *but* $|\mathcal{C}| = \mathbb{R} \Rightarrow |\mathcal{L}| = |\mathcal{P}(\mathbb{R})|$

Remark. $|\mathcal{P}(\mathbb{R}) \setminus \mathcal{L}| = |\mathcal{P}(\mathbb{R})|$*: Let* V *be a Vitali set,* $V[0,1]$*, then* $\forall A \subset [2,3], V \cup A \notin \mathcal{L}$ *and so* $|\mathcal{P}(\mathbb{R})| \geq |\mathcal{P}(\mathbb{R}) \setminus \mathcal{L}| \geq |\mathcal{P}([2,3])| = |\mathcal{P}(\mathbb{R})|$

Cantor-Lebesgue Function

Let $U_k \coloneqq [0,1] \setminus \mathcal{C}_k$, which is $2^k - 1$ disjoint open intervals, of various lengths, and

$$U = [0,1] \setminus \mathcal{C} = [0,1] \setminus \bigcap_{k=1}^{\infty} \mathcal{C}_k = \bigcup_{k=1}^{\infty} U_k$$

Thus U is open on $[0,1]$ and $m(U) = m([0,1]) = 1$ since $m(\mathcal{C}) = 0$.

Theorem 8.2. *There is a continuous (weakly) increasing function* $\phi : [0,1] \to [0,1]$ *that is surjective with* $\phi(0) = 0$ *and* $\phi(1) = 1$ *such that* ϕ *is differentiable in* U *and* $\phi'(x) = 0 \ \forall x \in U$

First define ϕ on U_k by setting it to be equal to the constants $\{\frac{1}{2^k}, \frac{2}{2^k}, \ldots, \frac{2^k-1}{2^k}\}$ on it's $2^k - 1$ open intervals. Observe that if we increase $k \to k+1$, U_{k+1} has more intervals but some of them are the same that we already had in U_k, and on those, the value of ϕ in the 2 steps agrees!

Taking the union over k defines ϕ on U. To extend ϕ to all of $[0,1]$, we let $\phi(0) = 0$ and for all $x \in \mathcal{C} \setminus \{0\}$ let $\phi|x| \coloneqq \sup\{\phi(y) \mid y \in U \cap [0,x)\}$ (this is finite since ≤ 1)

We have defined a function $\phi : [0,1] \to [0,1)$ and it satisfies the specified properties.

Consider now $\psi(x) \coloneqq \phi(x) + x$ for $x \in [0,1]$. Some obvious properties:

- ψ is continuous
- ψ is strictly increasing
- $\psi(0) = 0$, $\psi(1) = 2$
- $\psi([0,1]) = [0,2]$ and ψ is a bijection between these
- $\psi^{-1} : [0,2] \to [0,1]$ is continuous

Proposition. *$m(\psi(\mathcal{C})) = 1$ and $\exists E \subset \mathcal{C}$, $E \in \mathcal{L}$ such that $\psi(E) \notin \mathcal{L}$*

Corollary. *This set E is measurable but not Borel.*

Proposition (Continuity of Measure)**.** *1. If $\{A_j\}_{j=1}^{\infty}$ are measurable sets with $A_j \subset A_{j+1}$ $\forall j$, then*

$$m\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} m(A_j)$$

2. If $\{B_j\}_{j=1}^{\infty}$ are measurable sets with $B_{j+1} \subset B_j$ $\forall j$, and $m(B_j) < \infty \iff m(B_1) < \infty$ then

$$m\left(\bigcap_{j=1}^{\infty} B_j\right) = \lim_{j\to\infty} m(B_j)$$

Definition 8.1 (Almost Everywhere)**.** *We say some property "P" holds almost everywhere on E, or for a.e $x \in E$, if $\exists E_0 \subset E$ with $m^\star(E_0) = 0$ such that P holds for* all *$x \in E \setminus E_0$. We also say "P holds for almost all x in E".*

Ex: Almost every real number is irrational.

Proposition (Borel-Cantelli's Lemma)**.** *Let $\{E_j\}_{j=1}^{\infty} \subset \mathcal{L}$ be such that $\sum_{j=1}^{\infty} m(E_j) < \infty$. Then almost every $x \in \mathbb{R}$ belongs to at most finitely many E_j's.*

Proof. For each n,

$$m\left(\bigcup_{j=n}^{\infty}\right) \leq \sum_{j=n}^{\infty} m(E_j) < \infty$$

and

$$\bigcup_{j=n+1}^{\infty} E_j \subset \bigcup_{j=n}^{\infty} E_j$$

So by the continuity of measure

$$m\left(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty} E_j\right) = \lim_{n\to\infty} m\left(\bigcup_{j=n}^{\infty} E_j\right) \leq \lim_{n\to\infty}\sum_{j=n}^{\infty} m(E_j) \underbrace{=}_{\text{tails of a convergent series}} 0$$

Hence "almost every" $x \in E$ satisfies $x \notin \cap_{n=1}^{\infty} \cup_{j=n}^{\infty} E_j$. i.e for each such x, $\exists n$ such that $x \notin \cup_{j=n}^{\infty} E_n$ so x belongs only to (at most) $E_1 \dots E_{n-1}$ $\square$

9 Measurable Functions

We shall now study functions $f : E \to [-\infty, \infty] := \mathbb{R} \cup \{\pm\infty\}$ where $E \subset \mathbb{R}$ is a *measurable* set.

Sublevel sets of f are the sets of the form $f^{-1}([-\infty, c)) = \{x \in E \mid f(x) < c\}$, for some $c \in \mathbb{R}$

Definition 9.1. *If we have $f : E \to [-\infty, \infty]$ with E measurable, then we say that f is measurable if all sublevel sets $f^{-1}([-\infty, c))$ are in $\mathcal{L}$ for all $x \in \mathbb{R}$.*

Proposition. *$f : E \to [-\infty, \infty]$, then the following are equivalent:*

1. *f measurable*
2. *$\forall c \in \mathbb{R}$, $f^{-1}([-\infty, c]) = \{x \in E \mid f(x) \leq c\} \in \mathcal{L}$*
3. *$\forall c \in \mathbb{R}$, $f^{-1}((c, \infty]) = \{x \in E \mid f(x) > c\} \in \mathcal{L}$*
4. *$\forall c \in \mathbb{R}$, $f^{-1}([c, \infty]) \in \mathcal{L}$*
5. *$\forall U \subset \mathbb{R}$ open, $f^{-1}(U) \in \mathcal{L}$*

6. *$\forall A \subset \mathbb{R}$ Borel set, $f^{-1}(A) \in \mathcal{L}$*

Ex: If E measurable, $f : E \to \mathbb{R}$ continuous, then f is measurable. Indeed, $\forall U \subset \mathbb{R}$ open, $f^{-1}(U)$ is open in E, i.e $f^{-1}(U) = V \cap E$ where $V \subset \mathbb{R}$ open. Clearly $V \cap E \in \mathcal{L}$, so f is measurable.

Caution: $f : E \to \mathbb{R}$ continuous and $A \subset \mathbb{R}$ measurable $\not\Longrightarrow$ $f^{-1}(A) \in \mathcal{L}$. For example: $E = [0,1], f = \psi^{-1}$ then we proved earlier that ψ maps a measurable subset onto a non-measurable subset.

Proposition. *$f : [a,b] \to \mathbb{R}$ monotone $\implies$ f measurable*

Proof. without loss of generality, we may assume f is monotone increasing $f(x) \leq f(y)$ whenever $x \leq y$. For any $c \in \mathbb{R}$, look at $\{f < c\}$ and assume it is non-empty. We show that $\{f < c\}$ is an interval $\subset [a,b]$. Now, intervals $I \in \mathbb{R}$ are characterized by the property that if $x \leq y \in I$ then the whole segment $tx + (1-t)y$ is in I, for $0 \leq t \leq 1$. So let $f(x) < c$, $f(y) < c$, then $tx + (1-t)y \leq y$ so $f(tx + (1-t)y) \leq f(y) < c$ too.

So $\{f < c\}$ is an interval which means that f is measurable. $\square$

Proposition. *given $E \subset \mathbb{R}$ measurable, $f : E \to [-\infty, \infty]$ measurable*

1. *If $g : E \to [-\infty, \infty]$ is another function and $f = g$ a.e on E. Then g is measurable*
2. *Suppose $D \subset E$, D measurable. Then f is measurable (as a function on E) $\iff$ $f\mid_D$ measurable (as a function on D) and $f\mid_{E\setminus D}$ is measurable (as a function on $E \setminus D$).*

Proof. (1): Let $A = \{x \in E \mid f(x) \neq g(x)\}$, which by assumption has $m(A) = 0$. Then $\forall c \in \mathbb{R}$,

$$\begin{aligned}\{x \in E \mid g(x) > c\} &= \{x \in A \mid g(x) > c\} \cup \{x \in E \setminus A \mid f(x) > c\} \\ &= \{x \in A \mid g(x) > c\} \cup \underbrace{\{x \in E \mid f(x) > c\}}_{\in\mathcal{L}} \cap \underbrace{\{E \setminus A\}}_{\in\mathcal{L}}\end{aligned}$$

$\{x \in A \mid g(x) > c\}$ is a subset of A hence it has measure 0 and is also measurable so $\{g > c\} \in \mathcal{L}$.

(2):

$$\begin{aligned}\{x \in E \mid f(x) > c\} &= \{x \in D \mid f(x) > c\} \cup \{x \in E \setminus D \mid f(x) > c\} \\ &= (\{x \in E \mid f(x) > c\} \cap D) \cup (\{x \in E \mid f(x) > c\} \cap (E \setminus D))\end{aligned}$$

$\square$

Sums and Products: If $f, g : E \to [-\infty, \infty]$ can we consider their sum $f+g$? Well, if $f(x) = \infty$ and $g(x) = -\infty$ then $f(x) + g(x)$ is definitely undefined. Let us then assume that f and g are finite for a.e point in E. Thus, $\exists E_0 \subset E$ with $m(E_0) = 0$, such that f and g are finite on $E \setminus E_0$. We will now show that $f+g : E \setminus E_0 \to \mathbb{R}$ is measurable (on $E \setminus E_0$). Then if $h : E \to [-\infty, \infty]$ is any function such that $h\,|_{E\setminus E_0} = (f+g)\,|_{E\setminus E_0}$ then h is also measureable by part (2) above. Observe that such an h always exists (e.g set $h = f+g$ on $E \setminus E_0$ and $h = 0$ on E_0), and it is not unique at all. However, as we just said, all such h are measurable. We thus can say $f+g$ is measurable on E.

Proposition. *$f, g : E \to [-\infty, \infty]$ measurable such that f, g are finite a.e on E. Then $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ and fg are measurable on E.*

However, composition of two measurable functions may fail to be measurable:

Ex: If $E \subset \mathbb{R}$ measurable let χ_E be its characteristic function

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Then χ_E is measurable on $\mathbb{R}$

$$\{\chi_E < c\} = \begin{cases} \mathbb{R} & c \geq 1 \\ E^c & 0 < c < 1 \\ \emptyset & c \leq 0 \end{cases}$$

Take then ψ from before, $\psi : [0,1] \to [0,2]$ strictly increasing, with $A \subset [0,1], A \in \mathcal{L}$ and $\psi(A) \notin \mathcal{L}$. Extend ψ to $\mathbb{R}$ as strictly increasing and continuous, for example with

$$\tilde{\psi}(x) = \begin{cases} \psi(x) & \text{if } 0 \leq x \leq 1 \\ x & \text{if } x < 0 \\ 2x & \text{if } x > 1 \end{cases}$$

So $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous bijection which implies $\tilde{\psi}^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous $\implies$ $\tilde{\psi}^{-1}$ measurable; χ_A is also measurable, but $f = \chi_A \circ \tilde{\psi}^{-1} : \mathbb{R} \to \mathbb{R}$ is *NOT* measurable, since if $I = \left(\frac{1}{2}, 2\right)$, $\chi_A^{-1}(I) = A$ then $f^{-1}(I) = \tilde{\psi}\left(\chi_A^{-1}(I)\right) = \tilde{\psi}(A) = \psi(A) \notin \mathcal{L}$.

To reconcile this, we introduce the following:

Proposition. *If $g : E \to \mathbb{R}$ is measurable and $f : \mathbb{R} \to \mathbb{R}$ continuous then $f \circ g : E \to \mathbb{R}$ measurable.*

Proof. $\forall U \subset \mathbb{R}$ open,

$$(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U)) \in \mathcal{L}$$

Since $f^{-1}(U)$ is open and g is measurable. □

Example: Take $f : E \to \mathbb{R}$ measurable, and $p \in \mathbb{R} > 0$. Then $|f|^p : E \to \mathbb{R}$ is measurable (indeed $y \to |y|^p$ is continuous on $\mathbb{R}$)

Proposition. *Let $\{f_j\}_{j=1}^n$ be a set of measurable functions $E \to \mathbb{R}$, then $\max_{1 \le j \le n}\{f_j\}$ and $\min_j\{f_j\}$ are measurable.*

Proof.

$$\{x \in E \mid \max_j\{f_j\}(x) > c\} = \bigcup_{j=1}^n \{x \in E \mid f_j(x) > c\}$$

While $\min\{f_j\} = -\max\{-f_j\}$ □

Convergence of functions: $\{f_n\}_{n=1}^\infty$, $f : E \to [-\infty, \infty], A \subset E$. We say that $f_n \to f$ as $n \to \infty$

1. Pointwise on A if $\forall x \in A \lim_{n\to\infty} f_n(x) = f(x)$
2. Pointwise a.e on A if $\exists B \subset \mathbb{R}$ such that $m(B) = 0$ and $f_n \to f$ pointwise on $A \setminus B$
3. Uniformly on A if f_n, f are $\mathbb{R}$-valued and $\forall \epsilon > 0\ \exists n_0$ such that $\forall x \in A$, $|f_n(x) - f(x)| \le \epsilon$, for all $n \ge n_0$.

Clearly $(c) \implies (b) \implies (a)$ but the reverse arrows are all false. For example $f_n(x) = x^n \to 0$ pointwise a.e on [0,1] but not pointwise on [0,1], and $f_n(x) = \sin(\frac{x}{n}) \to 0$ converges pointwise on $\mathbb{R}$ but not uniformly.

Proposition. *If $E \in \mathcal{L}$ and $f, f_n : E \to [-\infty, \infty]$ with all f_n being measurable and $f_n \to f$ pointwise on E, the f is measurable.*

Definition 9.2 (Simple Function)**.** *If E measurable, then $\psi : E \to \mathbb{R}$ is called simple if it is measurable, and takes only a finite number of values. Call these values $\{c_j\}_{j=1}^n$, for some $n \geq 1$. Then if we call $E_j = \psi^{-1}(c_j) = \{x \in E \mid \psi(x) = c_j\}$ then we have E_j measurable $\forall j = 1 \ldots n$ and $E = \cup_{j=1}^n E_j$ disjoint. Also $\psi = c_j$ on E_j so*

$$\boxed{\psi = \sum_{j=1}^{n} \chi_E c_j}$$

In other words, simple functions are the same thing as finite linear combinations (with $\mathbb{R}$ coefficients) of characteristic functions of measurable sets.

Approximation Lemma: We have E measurable and $f : E \to \mathbb{R}$ measurable. Suppose f is bounded, i.e $\exists C > 0$ such that $|f| \leq C$ then $\forall \epsilon\ \exists \phi_\epsilon, \psi_\epsilon$ simple functions on E such that $\phi_\epsilon \leq f \leq \psi_\epsilon$ on E and $0 \leq \psi_\epsilon - \phi_\epsilon \leq \epsilon$ on E.

Proposition. *$E \subset \mathbb{R}$ measurable, $f : E \to [-\infty, \infty]$. Then f is measurable $\iff \exists \{\psi_n\}_{n=1}^\infty$, $\psi_n : E \to \mathbb{R}$ simple functions, $\psi_n \to f$ pointwise on E, and $|\psi_n| \leq |f|$ on E, for all n. If $f \geq 0$, we may choose ψ_n such that $\psi_{n+1} \leq \psi_n$ on E $\forall n$.*

Definition 9.3 (Null-Set)**.** *A set $A \subset \mathbb{R}$ with $m^\star(A) = 0$ is called a null-set.*

Theorem 9.1 (Egorov's Theorem)**.** *For $E \in \mathcal{L}$ with $m(E) < \infty$, Let $\{f_n\}_{n=1}^\infty$ be measurable functions. $f_n : E \to [-\infty, \infty]$ which converge pointwise a.e to $f : E \to [-\infty, \infty]$ which is finite a.e on E (i.e f is $\mathbb{R}$-valued except for a null-set in E). Then $\forall \epsilon > 0$, $\exists F \subset E$ closed set, such that $m(E \setminus F) \leq \epsilon$ and $f_n \to f$ uniformly on F.*

To start, observe that we may assume there are $E_0, E_0' \subset E$ two null sets such that $f_n \to f$ pointwise on $E \setminus E_0$ and $f : E \setminus E_0' \to \mathbb{R}$. Thus, both of these hold on $E \setminus (\underbrace{E_0 \cup E_0'}_{\text{still a null set}})$, and if we prove Egorov on $E \setminus (E_0 \cup E_0')$ then this gives Egorov on E. Thus, up to relabeling $E \rightsquigarrow E \setminus (E_0 \cup E_0')$, we shall assume form the start that

$$\boxed{f_n \to f \text{ pointwise on } E \text{ and } f : E \to \mathbb{R}}$$

We already know that f is measurable on E.

Lemma 9.2. *Suppose we are in this setting. Then, $\forall \eta > 0$, $\forall \delta > 0$, $\exists A \subset E, A \in \mathcal{L}$, and $\exists N \geq 1$ such that $m(E \setminus A) \leq \delta$ and $|f_n - f| \leq \eta$ on A for all $n \geq N$.*

Theorem 9.3 (Lusin's Theorem)**.** *Let $E \in \mathcal{L}$, $f : E \to [-\infty, \infty]$ be measurable and finite a.e, then $\forall \epsilon > 0$, $\exists F \subset E$ closed with $m(E \setminus F) \leq \epsilon$ and $\exists g : \mathbb{R} \to \mathbb{R}$ continuous, such that $f = g$ on F.*

10 Integration

Definition 10.1 (Step Functions)**.** *Step functions are a special class of simple functions. $\phi : [a, b] \to \mathbb{R}$ is a step function if there exist finitely many disjoint intervals $\{E_j\}_{j=1}^n$, $E_j \subset [a, b] \forall j$, $\cup_{j=1}^n E_j = [a, b]$, and $\exists c_j \in \mathbb{R}$, such that $\phi = \sum_{j=1}^n c_j \chi_{E_j}$.*

Observe that if ϕ is a step function then $\{E_j\}_{j=1}^n$ give us a partition $\mathcal{P}$ of $[a, b]$ and

$$\mathbf{L}(\phi, \mathcal{P}) = \sum_{j=1}^n c_j \ell(E_j) = \mathbf{U}(\phi, \mathcal{P})$$

Where $\mathbf{L}$ and $\mathbf{U}$ are the lower Darboux sums defined in Riemann integration. So for any partition $\mathcal{Q}$ of [a,b]

$$\sup_{Q} \mathbf{L}(\phi, \mathcal{Q}) \geq \mathbf{L}(\phi, \mathcal{P}) = \mathbf{U}(\phi, \mathcal{P}) \geq \inf_{\mathcal{Q}} \mathbf{U}(\phi, \mathcal{Q}) \geq \sup_{Q} \mathbf{L}(\phi, \mathcal{Q})$$

Hence they are equal, and ϕ is Riemann integrable and

$$\int_a^b \phi(x) dx = \sum_{j=1}^n c_j \ell(E_j)$$

One can prove that if f is Riemann integrable on $[a, b]$, then

$$\begin{aligned}&\sup\left\{\int_a^b \phi(x)dx \mid \phi \text{ step function and } \phi \le f \text{ on } [a,b]\right\}\\ &= \inf\left\{\int_a^b \psi(x)dx \mid \psi \text{ step function and } \psi \ge f \text{ on } [a,b]\right\}\end{aligned}$$

To define the *Lebesgue Integral* we will proceed in steps.

Step 1:

Suppose ϕ is a simple function, so $E \in \mathcal{L}$, $\phi : E \to \mathbb{R}$ has the form $\phi = \sum_{j=1}^n a_j \chi_{E_j}$ where $a_j \in \mathbb{R}$ is distinct and $E_j \subset E$, $\cup_{j=1}^n E_j = E$ is a disjoint union.

Suppose $m(E) < \infty$, then we define the Legesbue integral as

$$\boxed{\int_E \phi = \int_E \phi(x)dx = \sum_{j=1}^n a_j m(E_j)}$$

Proposition. *$E \in \mathcal{L}$ with $m(E) < \infty$, $\phi, \psi : E \to \mathbb{R}$ are simple functions then $\forall \alpha, \beta \in \mathbb{R}$,*

$$\int_E \alpha\phi + \beta\psi = \alpha \int_E \phi + \beta \int_E \psi \qquad \text{(Linearity)}$$

Also, if $\phi \le \psi$ on E, then

$$\int_E \phi \le \int_E \psi \qquad \text{(Monotonicity)}$$

Step 2:

$E \in \mathcal{L}$, $m(E) < \infty$, $f : E \to \mathbb{R}$ bounded. We say that f is Lebesgue integrable if $\mathbf{L}(f) = \mathbf{U}(f)$ where

$$\begin{aligned}\mathbf{L}(f) &= \sup\left\{\int_a^b \phi(x)dx \mid \phi \text{ step function and } \phi \le f \text{ on } [a,b]\right\}\\ \mathbf{U}(f) &= \inf\left\{\int_a^b \psi(x)dx \mid \psi \text{ step function and } \psi \ge f \text{ on } [a,b]\right\}\end{aligned}$$

Theorem 10.1. *$a, b \in \mathbb{R}$, $a < b$, $f : [a, b] \to \mathbb{R}$ a bounded function. Suppose f is Riemann integrable, then f is Lebesgue integrable on $[a, b]$ and the two integrals are equal.*

Theorem 10.2. *$E \in \mathcal{L}$ with $m(E) < \infty$, $f : E \to \mathbb{R}$ measurable and bounded, then f is Lebesgue integrable over E.*

Theorem 10.3. *$E \in \mathcal{L}, m(E) < \infty$, $f, g : E \to \mathbb{R}$ bounded measurable functions. $\forall \alpha, \beta \in \mathbb{R}$,*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$$

Also, if $f \leq g$ on E then $\int_E f \leq \int_E g$

Corollary 10.3.1 (Chopping). *$E \in \mathcal{L}$, $m(E) < \infty$, $f : E \to \mathbb{R}$ bounded and measurable. If $A, B \subset E$, $A, B \in \mathcal{L}$, $A \cap B = \emptyset$, then*

$$\boxed{\int_{A \cup B} f = \int_A f + \int_B f}$$

Proposition (Extremely Useful Inequality). *$E \in \mathcal{L}$, $m(E) < \infty$, $f : E \to \mathbb{R}$ bounded and measurable, then*

$$\boxed{\left| \int_E f \right| \leq \int_E |f|}$$

Proposition. *$E \in \mathcal{L}$, $m(E) < \infty$, $f_n : E \to \mathbb{R}$ bounded measurable. If $f_n \to f$ <u>uniformly</u> on E, then*

$$\lim_{n \to \infty} \int_E f_n = \int_E f$$

Theorem 10.4 (Bounded Convergence Theorem). *$E \in \mathcal{L}$, $m(E) < \infty$, $f_n : E \to \mathbb{R}$ bounded, $f_n \to f$ pointwise on E. Suppose that $\exists M > 0$ such that $|f_n| \leq M$ on E, $\forall n$. Then*

$$\lim_{n \to \infty} \int_E f_n = \int_E f$$

<u>Step 3:</u>

Definition 10.2 (Finite Support)**.** *$E \in \mathcal{L}$, not necessarily with $m(E) < \infty$. $f : E \to [-\infty, \infty]$ measurable. We say that f has* finite support *if its support $Supp(f) = \{x \in E : f(x) \neq 0\} \in \mathcal{L}$ satisfies $m(Supp(f)) <$. In other words, f is zero outside a measurable subset with finite measure. In this case, if $f : E \to \mathbb{R}$ bounded and measurable, $m(E)$ may be infinite, and if f has finite support, we define*

$$\int_E f := \int_{Supp(f)} f$$

Now, for $E \in \mathcal{L}$ and $f : E \to [0, \infty]$ measurable non-negative function, define

$$\int_E f = \sup \left\{ \int_E h \mid h : E \to \mathbb{R} \text{ bounded, measurable of finite support with } 0 \leq h \leq f \text{ on } E \right\}$$

Theorem 10.5 (Chebyshev's Inequality)**.** *$E \in \mathcal{L}$, $f : E \to [0, \infty]$ measurable. Then $\forall \lambda > 0$.*

$$\boxed{m\{f \geq \lambda\} \leq \frac{1}{\lambda} \int_E f}$$

Corollary 10.5.1. *$E \in \mathcal{L}$, $f : E \to [0, \infty]$ measurable, then*

$$\int_E f = 0 \iff f = 0 \text{ a.e on } E$$

Linearity and Monotonicity also apply to step 3 of the definition.

Proposition (Fatou's Lemma)**.** *$E \in \mathcal{L}$, $f_n : E \to [0, \infty]$ measurable, suppose $f_n \to f$ pointwise a.e on E. Then*

$$\int_E f \leq \liminf_{n \to \infty} \int_E f_n$$

Theorem 10.6 (Monotone Convergence Theorem)**.** *$E \in \mathcal{L}$, $f_n : E \to [0.\infty]$ measurable with $\{f_n\}$ increasing (i.e $f_n \leq f_{n+1}$ on E $\forall n \geq 1$). Assume $f_n \to f$ pointwise a.e on E. Then*

$$\boxed{\int_E f = \lim_{n \to \infty} \int_E f_n}$$

Definition 10.3. *$E \in \mathcal{L}$, $f : E \to [0, \infty]$ measurable. We say that f is <u>integrable</u> over E if $\int_E f < \infty$.*

Proposition. *f integrable $\implies$ f finite a.e on E.*

Proposition (Beppolevi's Lemma)**.** *$E \in \mathcal{L}$, $f_n : E \to [0,\infty]$ measurable with $f_n \leq f_{n+1}$ $\forall n$. Suppose $\exists C > 0$ such that $\int_E f_n \leq C$ $\forall n$. Then $f_n \to f$ pointwise on E, $f : E \to [0,\infty]$ measurable and finite a.e on E, and $\lim_{n\to\infty} \int_E f_n = \int_E f < \infty$.*

Step 4:

Now for general functions. $E \in \mathcal{L}, f : E \to [-\infty,\infty]$, measurable. Then $f^+, f^- : E \to [0,\infty]$ are measureable and

$$\begin{cases} f & = f^+ - f^- \\ |f| & = f^+ + f^- \end{cases}$$

on E.

Lemma 10.7. *$|f|$ integrable on E $\iff$ f^+ and f^- integrable on E*

Definition 10.4. *$E \in \mathcal{L}, f : E \to [-\infty,\infty]$ measureable. We say that f is integrable if $|f|$ integrable. Then let*

$$\int_E f = \int_E f^+ - \int_E f^- \in \mathbb{R}$$

This clearly agrees with the earlier definition if $f \leq 0$, since then $f^- = 0$.

Proposition. *f integrable on E $\implies$ f finite a.e on E, and $\forall E_0$ null set in E,*

$$\int_E f = \int_{E \setminus E_0} f$$

Proposition. *$E \in \mathcal{L}$, $f : E \to [-\infty,\infty]$ measurable. Suppose $g : E \to [0,\infty]$ measurable such that g integrable on E and $|f| \leq g$ on E. Then f also integrable, and*

$$|\int_E f| \leq \int_E |f|$$

Now if f, g are integrable over E, $f + g$ can only be defined at points where f and g are finite. But we know that $E_0 = \{f = \pm\infty\} \cup \{g = \pm\infty\}$ is a null set, so on $E \setminus E_0$ we define $f+g$, we will show that $f+g$ is integrable on $E \setminus E_0$, and then define $\int_E (f+g) \coloneqq \int_{E\setminus E_0}(f+g)$.

Linearity, monotonicity, and chopping also hold true for this definition of the Lebesgue integral.

Theorem 10.8 (Dominated Convergence)**.** *$E \in \mathcal{L}$, $f_n : E \to [-\infty, \infty]$ measurable. Suppose $f_n \to f$ pointwise a.e on E and $|f_n| \leq g$ on E $\forall n$ for some g integrable on E. Then f integrable and*

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

11 Lebesgue Measure in $\mathbb{R}^n$

We now briefly extend the theory of Lebesgue measure to $\mathbb{R}^n$, $n \geq 1$. To start, using open sets in $\mathbb{R}^n$, one defines the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}^n$.

To define the Lebesuge outer measure, the role of intervals is played by rectangles (or boxes). A box I in $\mathbb{R}^n$ is a product of intervals $I = I_1 \times \ldots \times I_n$ where each $I_j \subset \mathbb{R}$ is an interval. Then I open $\iff$ I_j open $\forall j$, I bounded $\iff$ I_j bounded $\forall j$.

The analogue of the length $\ell(I)$ is now the Volume $\mathrm{Vol}(I) = \prod_{j=1}^n \ell(I_j) \in [0,\infty]$. Clearly $\mathrm{Vol}(I) < \infty \iff I$ bounded.

If $A \subset \mathbb{R}^n$, let $\mathcal{C}_A = \left\{ \{I_j\}_{j=1}^\infty \mid I_j \text{ bounded open boxes with } A \subset \cup_{j=1}^\infty I_j \right\}$

Again, we let

$$m^\star(A) \coloneqq \inf_{\{I_j\}\in\mathcal{C}_A} \sum_{j=1}^\infty \mathrm{Vol}(I_j)$$

All of the classic properties of the case when $\mathbb{R} = 1$ also hold for $\mathbb{R}^n$.

Product Sets:

Lemma 11.1. *$A \subset \mathbb{R}^a$, $B \subset \mathbb{R}^b$ any sets, $A \times B \subset \mathbb{R}^{a+b}$, then $m^\star(A\times B) \leq m^\star(A)m^\star(B)$, with the convention that* $0 \times \infty = 0$

Proposition. *If $A \subset \mathbb{R}^a$, $A \in \mathcal{L}$, $B \subset \mathbb{R}^b$, $B \in \mathcal{L}$, then $A \times B \subset \mathbb{R}^{a+b}$ is measurable.*

Definition 11.1 (Slices)**.** *$E \subset \mathbb{R}^n$, $(n \geq 2)$, suppose $E \in \mathcal{L}$. A slice of E is a set of this form: write $\mathbb{R}^n = \mathbb{R}^a \times \mathbb{R}^b$, $a + b = n$.*

Pick any $x \in \mathbb{R}^a$ and let $E_x =$ slice $= \{y \in \mathbb{R}^b \mid (x, y) \in E\}$. There is a problem: $E \in \mathcal{L} \not\Rightarrow E_x \in \mathcal{L}$

Theorem 11.2 (Fubini's Theorem)**.** *Suppose $f : \mathbb{R}^n = \mathbb{R}^a \times \mathbb{R}^b \to [-\infty, \infty]$ is integrable with respect to Lebesgue on $\mathbb{R}^n$. Then for a.e $y \in \mathbb{R}^b$, the slice $f(\cdot, y)$ is integrable in $\mathbb{R}^a$ and the function $y \mapsto \int_{\mathbb{R}^a} f(x, y)dx$ is integrable in $\mathbb{R}^b$, we also have*

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^b} \left(\int_{\mathbb{R}^a} f(x, y)dx \right) dy$$

The theorem is symmetric in x and y so we also have $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^a} \left(\int_{\mathbb{R}^b} f(x, y)dy \right) dx$ and for a.e $x \in \mathbb{R}^a$, $f(x, \cdot)$ is integrable in $\mathbb{R}^b$ and $x \mapsto \int_{\mathbb{R}^b} f(x, y)dy$ is integrable in $\mathbb{R}$.

Corollary 11.2.1 (Tonelli's Theorem)**.** *$f : \mathbb{R}^n = \mathbb{R}^a \times \mathbb{R}^b \to [0, \infty]$ is measurable nonegative function. Then for a.e $y \in \mathbb{R}^b$, $f(\cdot, y)$ is measurable on $\mathbb{R}$ and $y \mapsto \int_{\mathbb{R}^a} f(x, y)dx$ is measurable on $\mathbb{R}^b$, and*

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^b} \left(\int_{\mathbb{R}^a} f(x, y)dx \right) dy$$

Usually, one applies Tonelli to $|f|$, where f is measurable on $\mathbb{R}^n$, so that $\int_{\mathbb{R}^n} |f| = \int_{\mathbb{R}^b} \left(\int_{\mathbb{R}^a} |f|(x, y)dx \right) dy$, so if the LHS is finite, so is the RHS, hence f is integrable in $\mathbb{R}^n$, so Fubini applies to f,

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^b} \left(\int_{\mathbb{R}^a} f(x, y)dx \right) dy$$

Corollary 11.2.2 (Cavalieri's Formula)**.** *$E \subset \mathbb{R}^n = \mathbb{R}^a \times \mathbb{R}^b$ measurable, then for a.e $y \in \mathbb{R}^b$, E_y is measurable in $\mathbb{R}^a$. Also $y \mapsto m(E_y)$ is a measurable function and*

$$\boxed{m(E) = \int_{\mathbb{R}^b} m(E_y)dy}$$

Corollary 11.2.3. *If $A \subset \mathbb{R}^a$, $A \in \mathcal{L}$, $B \subset \mathbb{R}^b$, $B \in \mathcal{L}$, then $A \times B \subset \mathbb{R}^{a+b}$ is measurable (we already knew that) and $m(A \times B) = m(A)m(B)$.*