Math 475: Partial Differential Equations

Final Exam Review: Definitions and Theorems

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Contents

1. Pre-Requisite Material

1.1. Vector Cal

Definition 1.1 (Green's First Identity (Integration by Parts in higher dimension))**.** Note that this requires continuity on $\partial(\Omega)$:

$$\int_\Omega v\Delta u dx = \int_{\partial\Omega} v\frac{\partial u}{\partial n} ds - \int_\Omega \nabla u \cdot \nabla v \tag{1}$$

This is commonly used in energy function type arguments.

1.2. Ordinary Differential Equations

Solving ODEs with Integrating Factors:
If we have an ODE of the form

$$\frac{dy}{dx}P(x) = Q(x)$$

Then, we can solve the ODE using the integrating factor I:

$$I := e^{\int P(x)dx}$$

Solving Constant Coefficient ODEs:
Given a second-order ODE of the form

$$ay'' + by' + cy = 0$$

the **characteristic equation** is $ar^2 + br + c$. The roots of the characteristic equation will give us the family of solutions to the ODE:

$$y_1 = e^{r_1 t}$$
$$y_2 = e^{r_2 t}$$

2. Introduction

2.1. Definitions

Definition 2.1 (kth order PDE)**.** A **PDE** is a mathematical relation involving partial derivatives, i.e., if $u : \mathbb{R}^n \to \mathbb{R}$, $x = (x_1, ..., x_n)$ is a real-valued function of several variables. Then, any **kth-order PDE** can be expressed as:

$$F(D^k u, ..., Du, u, x) = 0 \text{ in } \Omega \tag{2}$$

Definition 2.2 (Elliptic PDE)**.** A PDE is **elliptic** if $\forall x$, $\mathbf{A}(\mathbf{x})$ has non-zero eigenvalues, all of the same sign.

Definition 2.3 (Hyperbolic PDE)**.** A PDE is **hyperbolic** if $\forall x$, $\mathbf{A}(\mathbf{x})$ has non-zero eigenvalues, all except one of the same sign.

Definition 2.4 (Parabolic PDE)**.** A PDE is **parabolic** if $\forall x$, $\mathbf{A}(\mathbf{x})$ has at least one zero eigenvalue.

3. Diffusion

3.1. Solving the Heat Equation on $\mathbb{R}$

Theorem 3.1 (Separation of Variables)**.** Assume that we are working with the heat equation in $\mathbb{R} \times [0, \infty[$ and that our PDE is *homogeneous*. Use the following Ansatz: $\exists$ separated solutions of the form $u(x,t) = X(x)T(t)$. This separation gives us two ODES to solve:

$$T'(t) = -k\lambda^2 T(t)$$
$$X''(x) = -\lambda^2 X(x)$$

Then, the solution's structure will be based off the sign of λ^2:

(i) If $\lambda^2 = 0$:

$$u(x,t) = Ax + B, \ A, B \in \mathbb{R} \tag{3}$$

(ii) If $\lambda^2 > 0$:

$$u(x,t) = e^{-k\lambda^2 t}\left[A\cos(\lambda x) + B\sin(\lambda x)\right] \tag{4}$$

(iii) If $\lambda^2 < 0$:

$$u(x,t) = e^{-k\lambda^2 t}[Ae^{|\lambda|x} + Be^{-|\lambda|x}] \tag{5}$$

Theorem 3.2 (Uniqueness for the BVP). If there exists at least one solution $u \in C^{(2,1)}(\bar{\Omega_T})$ to

$$(*) := \begin{cases} u_t - k\Delta u = 0 \text{ in } \Omega_T \\ u(x,0) = g(x) \text{ in } \Omega \\ + \text{ boundary conditions} \end{cases} \tag{6}$$

Then this solution is unique.

3.2. Max Principles

Definition 3.1 (Sub/Super Solution). We say that u is a **subsolution** to the heat equation if

$$u_t - k\Delta u \leq 0 \tag{7}$$

and we say that u is a **supersolution** to the heat equation if:

$$u_t - k\Delta u \geq 0 \tag{8}$$

Theorem 3.3 (Weak Max Principle). Suppose that $u \in C^{(2,1)}(\Omega_T) \cap C(\partial_p\Omega_T)$ and that u is a sub-solution, i.e.:

$$u_t - k\Delta u = q(x,t) \leq 0$$

then:

$$\max_{\overline{\Omega_T}} u \leq \max_{\partial_p\Omega_T} u$$

Similarly, if $q(x,t) \geq 0$ (i.e., u is a supersolution), then:

$$\min_{\overline{\Omega_T}} u \geq \min_{\partial_p\Omega_T} u$$

(Optimising over the edge is the same as optimising over the whole thing). In particular, if $u_t. - k\Delta u = 0$, then, we can bound the solution as:

$$\min_{\partial_p\Omega_T} u \leq u(x,t) \leq \max_{\partial_p\Omega_T} u \tag{9}$$

The temperature on the interior is bounded the temperature on the boundary.

Corollary 3.1 (Comparison and Stability). Let $v, w \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T})$ be solutions of

$$v_t - k\Delta v = f_1 \text{ and } w_t - k\Delta w = f_2$$

with f_1, f_2 bounded in Ω_T. Then:

(i) **(Comparison Principle)**. If $f_1 \leq f_2$ in Ω_T and if $v \leq w$ on $\partial_p\Omega_T$, then, $v \leq w$ in Ω_T (ordering on the boundary is preserved on the interior).

(ii) **(Stability)**. The following stability estimate holds:

$$\max_{\overline{\Omega_T}} |v - w| \leq \max_{\partial_p\Omega_T} |v - w| + T\sup_{\overline{\Omega_T}} |f_1 - f_2|$$

Theorem 3.4 (Strong Max Principle). Let $u \in C^{(2,1)}(\Omega_T) \cap C(\partial_p\Omega_T)$, and assume that Ω is a connected domain. Assume that $u_t - k\Delta u \leq 0$ (i.e., u is a sub-solution). Let $M := \max_{\partial_p\Omega_T} u = \max_{\overline{\Omega_T}} u$. Then, if $u(x_1, t_1) = M$ for some $(x_1, t_1) \in \Omega_T$, then $u \equiv M$ in $\overline{\Omega_{t_1}}$. Similarly, for super-solutions; if v attains an interior min at (x_1, t_1), then $v \equiv m$ in $\overline{\Omega_{t_1}}$.

3.3. Fundamental Solution

Definition 3.2 (Fundamental Solution). The function:

$$\Gamma_D(x,t) := \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \tag{10}$$

is called the **Fundamental Solution to the Heat Equation** in $\mathbb{R}\times]0,\infty[$. In $\mathbb{R}^n$:

$$\Gamma_D(x,t) := \frac{1}{(4\pi Dt)^{\frac{n}{2}}} e^{-\frac{||x||^2}{4Dt}}$$

The solution is in C^∞. Great!

Definition 3.3 (Dirac Delta Distribution / Measure). $\delta : \mathbb{R} \to [0,\infty]$, the **Dirac Delta Distribution**, denoted by δ is the generalised function that acts on test functions as follows:

$$\delta[\varphi] = \varphi(0) \tag{11}$$

which is equivalent to:

$$< \delta, \varphi >= \int_{\mathbb{R}} \delta(x)\varphi(x)dx = \varphi(0) \tag{12}$$

Definition 3.4 (Convolution). A function $(f * g)$ is the **convolution** of f and g and is defined by:

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)dy = (g * f)(x) \tag{13}$$

Theorem 3.5 (Existence of Solutions to the Global Cauchy Problem). Assume that $\exists\, a, c > 0$ such that:

$$|g(x)| \leq ce^{ax^2}$$

$\forall x \in \mathbb{R}$. Consider the GCP:

$$(GCP) := \begin{cases} u_t - D\Delta u = 0 \text{ in } \mathbb{R}\times]0,\infty[\\ u(x,0) = g(x) \end{cases}$$

and let u be given by:

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4Dt}} g(y)dy$$

Let $T \leq \frac{1}{4aD}$. Then, the following properties hold:

(i) $\exists\, c_1, A > 0$ such that:

$$|u(x,t)| \leq c_1 e^{Ax^2} \;\; \forall (x,t) \in \mathbb{R}\times]0,T]$$

(ii) $u \in C^\infty(\mathbb{R}\times]0,T])$ and in the strip $\mathbb{R}\times]0,T]$:

$$u_t - D\Delta u = 0$$

(iii) Let $(x,t) \to (x_0, 0^+)$. Then, if g is continuous at x_0, then:

$$u(x,t) \to g(x_0)$$

Question 3.1 (pg. 34)**.** How does the Dirac delta behave in integrals?

Question 3.2 (pg. 36)**.** Solve:

$$\begin{cases} u_t - Du_{xx} = 0 \text{ in } \mathbb{R}\times]0,\infty[\\ u(x,0) = e^{-x} \end{cases}$$

Definition 3.5 (Error Function)**.** The **error function** is given by:

$$\text{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2} dr \tag{14}$$

Theorem 3.6 (Global Max Principle)**.** Let $z \in C^{(2,1)}(\mathbb{R}^n\times]0,T[)$ and let $z(x,t) \leq Ce^{a|x|^2}$ for some $C > 0$ in $\mathbb{R}^n\times]0,T[$. Then, if:

(i) $z_t - Dz_{xx} \leq 0$ and $z(x,t) \leq Ce^{ax^2}$

$$z(x,t) \leq \sup_{\mathbb{R}^n\times[0,T]} z(x,t) \leq \sup_{\mathbb{R}^n} z(x,0) \tag{15}$$

(ii) $z_t - Dz_{xx} \geq 0$ and $z(x,t) \geq -Ce^{ax^2}$

$$\inf_{\mathbb{R}^n} z(x,0) \leq \inf_{\mathbb{R}^n\times[0,T]} z(x,t) \leq z(x,t) \tag{16}$$

Corollary 3.2. Uniqueness for GCP The unique solution to

$$\begin{cases} u_t - D\Delta u = 0 \text{ in } \mathbb{R}^n \times [0,T] \\ u(x,0) = g(x) \end{cases}$$

with $|x(x,t)| \leq Ce^{a|x|^2}$ is given by $u(x,t) = (\Gamma(\cdot,t) * g)(x)$

Corollary 3.3 (Comparison and data $\leftrightarrow$ solution correspondences)**.** ($\to$ the ways we can control the solutions in terms of the data).

(i) (**Comparison**): If u is a subsolution and v is a supersolution with $u(x,0) \leq v(x,0)$ and $u(x,t) - v(x,t) \leq Ce^{a|x|^2}$, then:

$$u(x,t) \leq v(x,t) \tag{17}$$

(ii) (**Estimates**): If

$$\begin{cases} u_t - D\Delta u = f \text{ in } \mathbb{R}^n \times [0,T] \\ u(x,0) = g(x) \end{cases}$$

and if $u(x,t)$ is a Tychoff function, then:

$$t \inf_{\mathbb{R}^n\times[0,T]} f + \inf_{\mathbb{R}^n} g \leq u(x,t) \leq \sup_{\mathbb{R}^n} g + t \sup_{\mathbb{R}^n\times[0,T]} f \tag{18}$$

(iii) (**Stability**): If:

$$\begin{cases} u_t - D\Delta u = f_1 \\ u(x,0) = g_1 \end{cases}$$

and

$$\begin{cases} v_t - D\Delta v = f_2 \\ v(x,0) = g_2 \end{cases}$$

Then:

$$|u(x,t) - v(x,t)| \leq \sup_{\mathbb{R}^n} |g_1 - g_2| + T \sup_{\mathbb{R}^n\times[0,T]} |f_1 - f_2| \tag{19}$$

3.4. Non-Homogeneous Heat Equation

Theorem 3.7 (Duhamel's Method)**.** Given the non-homogeneous problem:

$$(*) := \begin{cases} u_t - D\Delta u = f(x,t) \text{ in } \mathbb{R}^n \times]0,\infty[\\ u(x,0) = g(x) \end{cases}$$

if $g(x)$ is a Tychoff function, and if f, ∇f, $D^2 f$, and f are all continuous in $\mathbb{R}^n \times [0,T[$ (so that the integral is finite). Then, u solves $(*)$, where :

$$u(x,t) := h(x,t) + q(x,t)$$

where $h(x,t)$ solves:

$$\begin{cases} h_t - D\Delta h = 0 \text{ in } \mathbb{R}^n \times]0,\infty[\\ h(x,0) = g(x) \end{cases}$$

and where

$$q(x,t) = \int_0^t v(x,t-s;s)ds$$

where $v(x,t;s)$ solves:

$$\begin{cases} v_t - D\Delta v = 0 \text{ in } \mathbb{R}^n \times]0,\infty[\\ v(x,0;s) = f(x,s) \end{cases}$$

and thus:

$$u(x,t) = \int_{\mathbb{R}^n} \Gamma_D(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Gamma_D(x-y,t-s)f(y,s)dyds \tag{20}$$

If, moreover, $g(x)$ is continuous at x, then:

$$\lim_{t\to\infty} u(x,t) = g(x)$$

4. Laplace's Equation

Theorem 4.1 (Uniqueness for Laplace's Equation)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a smooth and bounded domain. Then, there exists at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that solves

$$\begin{cases} -\Delta v = f \text{ in } \Omega \\ + \text{ boundary conditions} \end{cases}$$

In the case of Neumann boundary conditions, then two solutions will differ by at most a constant.

4.1. Harmonic Functions

Theorem 4.2 (Mean Value Property for Harmonic Functions)**.** Let u be harmonic in $\Omega \subseteq \mathbb{R}^n$. Then, for all balls $B_R(x) \subseteq \Omega$, the following mean value formulae hold:

$$u(x) = \frac{1}{|\partial B_R(x)|} \int_{\partial B_R(x)} u(\sigma)d\sigma$$
$$u(x) = \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y)dy$$

Theorem 4.3 (Characterisation of Harmonic Functions)**.** Let $u \in C(\Omega)$. If u satisfies a mean value property, then $u \in C^\infty(\Omega)$ and it is harmonic in Ω.

Theorem 4.4 (Max Principle and Strong Max Principle for Harmonic Functions)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and let $u \in C^2(\Omega) \cap C(\partial\Omega)$. If u satisfies a mean value property ($\iff$ it is harmonic), then:

(i) **Regular Max Principle**:

$$u(x) \le \max_{\overline{\Omega}} u(x) = \max_{\partial\Omega} u(x)$$

$$\min_{\partial\Omega} u(x) \le \min_{\overline{\Omega}} u(x) \le u(x)$$

(ii) **Strong Max Principle**. Further assume that Ω is path-connected. Then, if u attains an interior max, then u is constant on Ω. If Ω is not path connected, then u is constant on that path-connected component.

Corollary 4.1 (Comparison, Uniqueness, and Stability for the Harmonic Function)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and let $g \in C(\partial\Omega)$. Then, the problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

has at most one solution $u_g \in C^2(\Omega) \cap (\overline{\Omega})$ (**Uniqueness**). Moreover, let u_{g_1}, u_{g_2} be the solutions corresponding to the data $g_1, g_2 \in C(\partial\Omega)$. Then:

(i) (**Comparison**). If $g_1 \ge g_2$ on $\partial\Omega$, then:

$$u_{g_1} \ge u_{g_2} \text{ in } \Omega$$

(ii) (**Stability**). $\forall x \in \Omega$:

$$|u_{g_1}(x) - u_{g_2}(x)| \le \max_{\partial\Omega} |g_1 - g_2|$$

4.2. Fundamental Solution

Theorem 4.5 (Fundamental Solution for the Laplace Operator)**.** Given $-\Delta u = 0$, we looked for invariant properties of the Δ operator to construct a fundamental solution. Due to rotation and translation invariance, we were able to derive the following **fundamental solution for the Laplace operator**:

$$\Phi(x) := \begin{cases} \frac{-1}{2\pi} \log|x| & n = 2 \\ \frac{1}{\omega_n |x|^{n-2}} & n \ge 3 \end{cases} \tag{21}$$

Definition 4.1 (Newtonian Potential)**.** The convolution between Φ and f is called the **Newtonian Potential** of f:

$$u(x) = \int_{\mathbb{R}^{\nvDash}} \Phi(x-y) f(y) dy = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy \tag{22}$$

Theorem 4.6. Let $f \in C^2(\mathbb{R}^n)$ with compact support. Let $u := (\Phi * f)$. If we are in the case of $n \ge 3$, then u is the only solution of

$$-\Delta u = f \tag{23}$$

belonging to $C^2(\mathbb{R}^n)$ that vanishes at ∞. In the case of $n = 2$, u is the only solution of 23 up to a condition of $\pm\infty$, since it will be the only solution such that: [1]

$$u(x) \sim c \log|x| + o\left(\frac{1}{|x|}\right)$$

[1] $o\left(\frac{1}{|x|}\right)$ means that it is bounded by $C\frac{1}{|x|}$

4.3. Green's Functions

Theorem 4.7 (Representation Theorem)**.** Let $\Omega \in \mathbb{R}^n$ be a smooth and bounded domain. For every $x \in \Omega$, and for every $u \in C^2(\overline{\Omega})$, we can express u in terms of useful quantities:

$$u(x) = -\int_{\Omega} \Phi(x-y)\Delta u(y)dy + \int_{\partial\Omega} \Phi(x-\sigma)\partial_v u d\sigma - \int_{\partial\Omega} u(\sigma)\partial_v \Phi(x-\sigma)d\sigma \tag{24}$$

Definition 4.2 (Green's Functions)**. Green's Functions**, denoted $G(x, y)$, are scalar valued functions $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of a bounded domain Ω that solves the following

$$\begin{cases} -\Delta_y G(x,y) = \delta(x-y) & \text{in } \Omega \\ G(x,\sigma) = 0 & \text{in } \partial\Omega \end{cases}$$

This takes care of the boundary value compatibility, since the Green Function G can be written in the form $G(x, y) = \Phi(x - y) - \varphi(x, y)$, where φ, for a fixed $x \in \Omega$, solves the following Dirichlet Problem:

$$\begin{cases} -\Delta_y \varphi(x,y) = 0 & \text{in } \Omega \\ \varphi(x,\sigma) = \Phi(x-\sigma) & \text{on } \partial\Omega \end{cases}$$

Theorem 4.8 (Expressing Solutions via Green's Function)**.** The unique solution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

is given by

$$u(x) = \int_{\Omega} G(x,y)f(y)dy - \int_{\partial\Omega} \partial_v G(x,y)g(\sigma)d\sigma \tag{25}$$

Definition 4.3 (Poisson's Formula on a Ball)**.** Given the PDE:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

The formula that we use to compute the solution $u(x)$ is called **Poisson's Formula**:

(i) Poisson's Formula in $\mathbb{R}^3$:

$$u(x) = \frac{R^2 - |x|^2}{4\pi R} \int_{\partial B_R(0)} \frac{g(\sigma)}{|x-\sigma|^3} d\sigma \tag{26}$$

(ii) Poisson's Formula in $\mathbb{R}^n$:

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{g(\sigma)}{|x-\sigma|^n} d\sigma \tag{27}$$

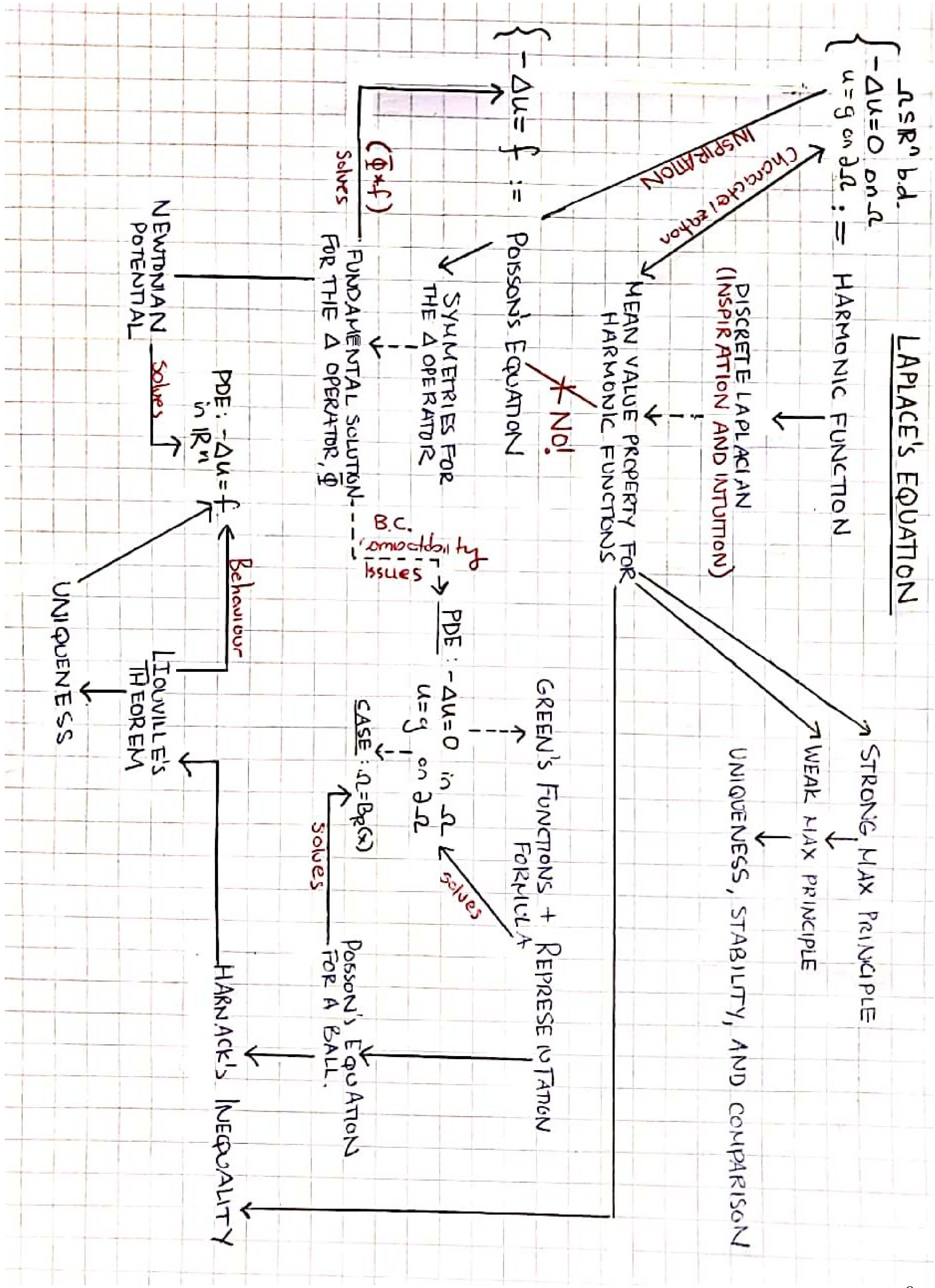
Theorem 4.9 (Harnack's Inequality)**.** If u is harmonic and non-negative in $\Omega \subseteq \mathbb{R}^n$, then $\forall K \subseteq \Omega$, K compact:

$$\min_K u \leq \max_K u \leq C \min_u u$$

where $C := C[n, \text{dist}(K,\ \partial\Omega)]$. Moreover, for any ball $B_R(z) \subset\subset \Omega$, then $\forall x \in B_R(z)$, define $r := |x - z|$, $r < R$. Then:

$$\frac{R^{n-2}(R-r)}{(R+r)^{n-1}} u(z) \leq u(x) \leq \frac{R^{n-2}(R+r)}{(R-r)^{n-1}} u(z)$$

Theorem 4.10 (Liouville's Theorem)**.** If u is harmonic and defined in all of $\mathbb{R}^n$ and is bounded from above or from below, then f is constant.


LAPLACE'S EQUATION
$\Omega \subseteq \mathbb{R}^n$ b.d.
$-\Delta u = 0$ on Ω
$u = g$ on $\partial\Omega$
:= HARMONIC FUNCTION
DISCRETE LAPLACIAN (INSPIRATION AND INTUITION)
MEAN VALUE PROPERTY FOR HARMONIC FUNCTIONS
INSPIRATION
Characterization
STRONG MAX PRINCIPLE
WEAK MAX PRINCIPLE
UNIQUENESS, STABILITY, AND COMPARISON
$-\Delta u = f$:= POISSON'S EQUATION
No!
SYMMETRIES FOR THE Δ OPERATOR
FUNDAMENTAL SOLUTION FOR THE Δ OPERATOR, Φ
$(\Phi * f)$
solves
NEWTONIAN POTENTIAL
solves
PDE: $-\Delta u = f$ in $\mathbb{R}^n$
B.C. compatibility issues
PDE: $-\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$
GREEN'S FUNCTIONS + REPRESENTATION FORMULA
solves
CASE: $\Omega = B_R(x)$
solves
POISSON'S EQUATION FOR A BALL
HARNACK'S INEQUALITY
LIOUVILLE'S THEOREM
Behaviour
UNIQUENESS

5. Scalar Conservation Laws and First Order Equations

Definition 5.1 (Transport in a Channel PDE)**.** Let $u : \mathbb{R} \times [0, \infty[\to \mathbb{R}$. Then, the transport of a concentrate u is modelled by:

$$\begin{cases} u_t + [q(u)]_x = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

Where

(i) $q(u)$ is the **flux function**. In the first derivation, $q(u) = uv$, which means that the flux is proportional to the amount of concentrate.
(ii) $u_0(x)$ is the initial concentration of the gas.
(iii) This is called a **first oder PDE in u**

Definition 5.2 (Scalar Conservation Law)**.** A solution u obeys a **scalar conservation law** if the following is true:

$$\frac{d}{dt} \int_{x_1}^{x_2} u(x, t) dx = -q(u(x_1, t)) - q(u(x_2, t)) \tag{28}$$

5.1. Characteristics

Definition 5.3 (Characteristic curve in $\mathbb{R}^2$)**.** A curve in $\mathbb{R}^2$ is called a **characteristic curve** if, at each point (x_0, y_0) of the curve, the tangent vector is given by:

$$\begin{bmatrix} a(x_0, y_0) \\ b(x_0, y_0) \end{bmatrix}$$

where $a(x, y)u_x + b(x, y)u_y = 0$.

Definition 5.4 (Integral Surface of the Vector Field)**.** Let $\mathbf{V} = (a(x, y, z), b(x, y, z), c(x, y, z))$ denote the vector field corresponding to the coefficients of a first order PDE. Then, the equation for the PDE tells us that $\mathbf{V}$ is tangent to the surface $(x, y, u(x, y))$. The graph of the solution (the height function $u(x, y)$) is called the **integral surface of the vector field V**.

Definition 5.5 (Characteristic Curves in the Case Where the Curves Are Parameterised)**.** Parameterise x with the variable τ as $x = x(\tau)$. The **characteristic curves** $x, y, z : I \subseteq \mathbb{R} \to \mathbb{R}$ are the curves that are parameterised by:

$$\frac{dx}{d\tau} = a(x, y, z)$$
$$\frac{dy}{d\tau} = b(x, y, z)$$
$$\frac{dz}{d\tau} = c(x, y, z)$$

The **projected characteristics** are given by $(x(\tau), y(\tau))$, and it turns out that $z(\tau) = y(x(\tau), y(\tau))$.

Theorem 5.1 (Method of Characteristics – Most General Case)**.** We are trying to solve the following PDE:

$$a(x, y, z)u_x + b(x, y, z)u_y - c(x, y, z) = 0$$

with a side-condition curve $u(x, y) = f(x, y)$. Then:

(i) Let $\mathcal{S}$ be the surface corresponding to $(x, y, u(x, y))$ with $u \in C^1$. If $\mathcal{S}$ is a union of characteristics, then u solves the PDE. Thus, solving the PDE $\leftrightarrow$ obtaining the surface.
(ii) Every integral surface of $\mathbf{V}$ is the *disjoint* union of characteristics. Thus, every point (x, y) belongs to exactly one characteristic curve.
(iii) (Uniqueness). If two characteristics intersect at one point, then they intersect everywhere.

The next family of theorems will give us sufficient, but not necessary conditions, for the Method of Characteristics to work.

Theorem 5.2 (Inverse Function Theorem)**.** Let $f : \Omega \to \mathbb{R}^n$ be C^k, and let $\Omega \subseteq \mathbb{R}^n$ be open. Let $x_0 \in \Omega$, $y_0 \in f(\Omega)$. Suppose that $\text{Jac}(f) = \det(Df(x_0)) \neq 0$. Then, there exists a small neighbourhood around (x_0, y_0) such that

(i) f is invertible.
(ii) the inverse of f is C^k.

Theorem 5.3 (Inverse Function Theorem $\Rightarrow$ Method of Characteristics Works)**.** Assume a, b, c are C^1 near (x_0, y_0, z_0) and $f, g, h \in C^1$ in $I \subseteq \mathbb{R}$. Then, if $\text{Jac}(s, \tau) = \text{Jac}(0, 0) \neq 0$ in a neighbourhood of (x_0, y_0), then there exists a unique solution u to the Cauchy Problem

$$\begin{cases} a(x, y, z)u_x + b(x, y, z)u_y = c(x, y, z) \\ u(f(s), g(s)) = h(s) \end{cases}$$

in a neighbourhood of (x_0, y_0, z_0). **CAUTION: Observe that this is not a characterisation of the method of characteristics, meaning that it is possible for the Inverse Function Theorem to fail but for there to still be a solution. Proving that there does not exist a solution requires showing a contradiction instead of showing that the conditions for the Inverse Function Theorem are not met.**

Theorem 5.4 (Implicit Function Theorem)**.** Let $\Omega \subseteq \mathbb{R} \times]0, \infty[$ be open. Suppose $F : \Omega \to \mathbb{R}$ is C^k. Suppose that:

(i) $F(u_0, x_0, 0) = 0$
(ii) $\frac{\partial F}{\partial u}(u_0, x_0, 0) \neq 0$

Then, there exists an open set $V \subseteq \Omega$ and $W \subseteq \mathbb{R}^2$ such that $(x_0, 0) \in W$ and a C^k function $u : W \to \mathbb{R}$ such that:

(i) $u(x_0, 0) = u_0(x_0)$
(ii) $F(y, x, t) = 0 \ \forall \ (x, t) \in W$.
(iii) If $(u, x, t) \in V$ and $F(u, x, t) = 0$, then $u = u(x, t)$.

Definition 5.6 (Formation of a Shock Curve)**.** Informally, a **shock curve** is formed when the characteristics cross. A shock curve is formed at the time $t = t^*$ when the derivative of the solution blows up. Bad.

Definition 5.7 (Weak / Entropy Solutions)**. Weak / Entropy solutions** are solutions that are used to extend solutions due to the discontinuity in the solution that a shock curve can cause

6. Waves and Vibrations

6.1. Waves and Vibrations in $\mathbb{R}$

Theorem 6.1 (Solving the Wave Equation in $\mathbb{R}$ with d'Alembert's Formula)**.** The unique solution to:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x, 0) = g(x) \\ u_t(x, 0) = h(x) \end{cases}$$

is

$$u(x,t) = \frac{1}{2}g(x-ct) + \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy$$

Theorem 6.2 (Max Estimate in $\mathbb{R}$)**.** Let $x \in \mathbb{R}$ and $t \in [0, \infty[$ and

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

Then:

(i) $|u(x,t)| \le \max_{x\in\mathbb{R}} |g(x)| + \frac{1}{2c} \max_{a<b} \left| \int_a^b h(y)dy \right|$ (the second term is a max over all intervals)

(ii) For all $T < \infty$ and $t \le T$:

$$|u(x,t)| \le \max_{x\in\mathbb{R}} |g(x)| + T \max_{x\in\mathbb{R}} |h(y)|$$

Corollary 6.1. The max estimate gives us **stability of solutions**. $\forall\ T < \infty$:

$$|u_1(x,t) - u_2(x,t)| \le \max_{x\in\mathbb{R}} |g_1(x) - g_2(x)| + T \max_{x\in\mathbb{R}} |h_1(y) - h_2(y)| \tag{29}$$

Definition 6.1 (Region of Influence)**.** Given x_0 at a time $t = 0$, the **Region of Influence** is the cone emanating from x_0 with slope of $\pm\frac{1}{c}$. It is the set of all points $(x_0, t) \in \mathbb{R}^2$ that x_0 can influence.

Definition 6.2 (Domain of Dependence)**.** The interval $[x_0 - ct_0, x + ct_0]$ is the **interval of dependence** for a point (x_0, t_0). It is the set of all points that can influence a point (x_0, y_0).

Definition 6.3 (Casualty Principle)**.** Only times before t_0 can influence the point (x_0, t_0).

Proposition 6.1 (Odd/Even Data $\Rightarrow$ Odd/Even Solutions)**.** The one-dimensional wave equation preserves the odd/evenness of data in the solution due to the behaviour of the second derivatives & uniqueness.

Proposition 6.2 (L-Periodic Data $\Rightarrow$ L-Periodic Solutions)**.** The one-dimensional wave equation preserves the L-periodicity of data in the solution due to the behaviour of the second derivatives & uniqueness.

Theorem 6.3 (Max Estimate for a Finite String)**.** Let u solve:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0;\ \ x \in [0, L],\ \ t \ge 0 \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \\ u(0,t) = u(L,t) = 0 \end{cases}$$

Then:

$$|u(x,t)| \le \max_{x\in[0,L]} |g(x)| + \frac{L}{c} \max_{x\in[0,L]} |h(x)| \tag{30}$$

Theorem 6.4 (Duhamel's Formula for the Wave Equation)**.** The solution for the inhomogeneous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,y) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

is given by:

$$u(x,t) = \int_0^t \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds \tag{31}$$

6.2. Multidimensional Wave Equation in $\mathbb{R}^3$

Proposition 6.3. If

$$\begin{cases} v_{tt} - c^2 \Delta v = 0 \quad x \in \mathbb{R}^n,\ t \geq 0 \\ v(x,0) = 0 \\ v_t(x,0) = g(x) \end{cases}$$

and if $v \in C^3$, then $u := v_t$ solves:

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 \quad x \in \mathbb{R}^n,\ t \geq 0 \\ u(x,0) = g(x) \\ u_t(x,0) = 0 \end{cases}$$

Theorem 6.5 (Fundamental Solution for the Wave Equation in $\mathbb{R}^3$)**.** Consider the Global Cauchy Problem

$$\begin{cases} w_{tt} - c^2 \Delta w = 0 \quad \text{in } \mathbb{R}^3 \times]0, \infty[\\ w(x,0) = 0 \\ w_t(x,0) = \delta(x) \end{cases}$$

then the function

$$k(x,t) := \frac{\delta(||x|| - ct)}{4\pi c ||x||} \tag{32}$$

solves the Global Cauchy Problem

Definition 6.4 (Region of Influence)**.** Since $K(x-y,t)$ is supported on $\{x \mid ||x-y|| = ct\}$, the **region of influence** is $\partial B(y, ct)$. It is the boundary of the three-dimensional cone emanating out of y. This phenomena demonstrated **Huygen's Principle**.

Theorem 6.6 (Kirchoff's Formula)**.** Let $g \in C^3(\mathbb{R}^3)$, $h \in C^2(\mathbb{R}^3)$. Then:

$$u(x,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{\partial B(x,ct)} g(\sigma) d\sigma\right] + \frac{1}{4\pi c^2 t}\int_{\partial B(x,ct)} h(\sigma) d\sigma \tag{33}$$

solves

$$\begin{cases} w_{tt} - c^2 \Delta w = 0 \quad \text{in } \mathbb{R}^3 \times]0, \infty[\\ w(x,0) = g(x) \\ w_t(x,0) = h(x) \end{cases}$$

6.3. Wave Equation in $\mathbb{R}^2$ – Hadamard's Method of Descent

Theorem 6.7 (Poisson's Formula)**.** The function

$$u(x,t) = \frac{1}{2\pi c}\int_{B(x,ct)} \frac{h(y)}{\sqrt{c^2t^2 - ||x-y||^2}} dy \tag{34}$$

solves

$$\begin{cases} w_{tt} - c^2 \Delta w = 0 \quad \text{in } \mathbb{R}^2 \times]0, \infty[\\ w(x,0) = 0 \\ w_t(x,0) = h(x) \end{cases}$$

7. Fourier Series

7.1. Definitions

Definition 7.1 (Fourier Series of f)**.** For a function $f(x)$, we define the **Fourier Series on** $[-L, L]$ by:

$$\text{FS } f(x) := \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \tag{35}$$

where we obtain the coefficients A_n and B_n by:

$$A_n := \frac{1}{L}\int_{[-L,L]} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \tag{36}$$

$$B_n := \frac{1}{L}\int_{[-L,L]} f(x)\sin\left(\frac{n\pi x}{L}\right) dx \tag{37}$$

Definition 7.2 (Inner Product)**.** Given two functions f, g on $[-L, L]$, the **inner product** of f and g is defined as:

$$< f, g >:= \int_{-L}^{L} f(x)g(x)fx \tag{38}$$

Note that this inner product is **bi-linear** since we are working in $\mathbb{R}$.

Definition 7.3 (L^2 norm)**.** The L^2 **norm** is defined as:

$$||f|| := (< f, f >)^{\frac{1}{2}} = \left(\int_{-L}^{L} [f(x)]^2 dx\right)^{\frac{1}{2}} \tag{39}$$

Definition 7.4 (Orthonormal Family of Functions)**.** We say that a sequence of functions $\{f_n\}_{n=1}^{\infty}$ form an **orthogonal family of functions** if

$$< f_m, f_n >= \begin{cases} 0; & m \neq m \\ 1; & m = n \end{cases} \tag{40}$$

Proposition 7.1. The family of functions

$$\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty} \tag{41}$$

forms an orthogonal family on $[-L, L]$. Similarly, the family of functions

$$\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty} \tag{42}$$

forms an orthogonal family on $[-L, L]$. Finally,

$$\left\{\frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right), \frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty} \tag{43}$$

Forms an orthonormal family on $[-L, L]$.

Theorem 7.1. Suppose that

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi x}{L}\right) \tag{44}$$

Then, the coefficients are obtained by:

$$A_n = \frac{1}{L}\left\langle f(x), \cos\left(\frac{n\pi x}{L}\right)\right\rangle \tag{45}$$

$$B_n = \frac{1}{L}\left\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle \tag{46}$$

Definition 7.5 (Pointwise Convergence)**.** Let $f : U \to \mathbb{R}$ and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions. We say that $f_n \to f$ **pointwise** if $\forall \varepsilon > 0,\ \forall x \in U,\ \exists\ N$ s.t. $\forall n \geq N$,

$$|f_n(x) - f(x)| \leq \varepsilon \tag{47}$$

Thus, for a fixed x we have a limit of a regular sequence of numbers in $\mathbb{R}$:

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{48}$$

Definition 7.6 (Uniform Convergence)**.** Let $f : U \to \mathbb{R}$ and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions. We say that $f_n \to f$ **uniformly** on U if $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t. $\forall\ n \geq N$:

$$\sup_{x\in U} |f_n(x) - f(x)| \leq \varepsilon \tag{49}$$

Definition 7.7 (L^2 convergence or mean square convergence)**.** Let $f : U \to \mathbb{R}$ and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions. We say that $f_n \to f$ in $\mathbf{L^2(U)}$ if $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t. $\forall\ n \geq N$:

$$\int_U |f_n(x) - f(x)|^2 dx \leq \varepsilon \tag{50}$$

Definition 7.8 (Convergence of Series)**.** Define the partial sums of a series of functions as:

$$s_N(x) := \sum_{n=1}^{N} f_n(x)$$

Then, the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges to $f(x)$...

(i) **Pointwise** if $s_N(x) \to f(x)$ pointwise.
(ii) **Uniformly** if $s_N(x) \to f(x)$ uniformly.
(iii) $\mathbf{L^2}$ if $s_N(x) \to f(x)$ in L^2.

Definition 7.9 ($L^2(U)$)**.** The function space $\mathbf{L^2(U)}$ is defined as:

$$L^2(U) := \left\{ f : U \to \mathbb{R} \mid \int_U |f(x)|^2 dx \leq \infty \right\} \tag{51}$$

Note that this creates an equivalence class of functions defined on $L^2(U)$. (Q: does it partition the space into functions that are equal a.e. on U?)

Theorem 7.2 (Bessel's Inequality)**.** Let $f \in L^2([-L, L])$. Then:

$$\frac{1}{2}A_0^2 + \sum_{n=1}^{\infty}(A_n^2 + B_n^2) \leq \frac{1}{L}\int_{-L}^{L}[f(x)]^2 dx \leq \infty \tag{52}$$

Theorem 7.3 (Riemann-Lebesgue Lemma)**.** If $f \in L^2([-L, L])$, then:

$$\lim_{n\to\infty}\left[\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx\right] = \lim_{n\to\infty} A_n = 0 \tag{53}$$

$$\lim_{n\to\infty}\left[\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx\right] = \lim_{n\to\infty} B_n = 0 \tag{54}$$

Theorem 7.4 (Pointwise convergence of Fourier Series)**.** Let $f \in C^1(]-L, L[)$. Then:

$$f(x) = \text{ FS } f(x) \tag{55}$$

converges pointwise in $]-L, L[$.

Theorem 7.5 (Uniform convergence of Fourier Series)**.** Let $f \in C^2(]-L, L[)$. Then:

$$f(x) = \text{ FS } f(x) \tag{56}$$

converges uniformly in $]-L, L[$.

Definition 7.10 (Fourier Sine and Cosine Series)**.** Let $f(x)$ be defined on $[0, L]$. Then:

(i) The **Fourier Cosine Series** F CS $f(x)$ is:

$$FS\ f_{\text{even}}(x) := \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \tag{57}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx \tag{58}$$

(ii) The **Fourier Sine Series** F SS $f(x)$ is:

$$FS\ f_{\text{odd}}(x) := \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \tag{59}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx \tag{60}$$