math 580

sam zeitler

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0.1 lecture 9/21

Theorem 0.1. *Poisson's Formula for the Ball*
Let $g \in C(\partial B_r(0))$ and v be defined by

$$v(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B_r(0)} \frac{g(y)}{|x-y|^n} dS = \int_{\partial B(0,r)} K(x,y)g(y)dS$$

Then

- $v \in C^\infty(B_r(0))$
- $-\Delta v = 0$ *in* $B_r(0)$
- *for all* $x_0 \in \partial B_r(0)$, $\lim_{x \to x_0} v(x) = g(x_0) \implies v \in C(\overline{B_r(0)})$

Proof. Recall $K(x,y) = -\frac{\partial G}{\partial \nu}$. We also showed

$$G(x,y) = G(y,x) \implies -\Delta_x G(x,y) = 0 \ \forall x \neq y$$

So if $x \in B_r(0)$, $y \in \partial B_r(0)$, then $-\Delta_x \frac{\partial G}{\partial \nu} = 0 = \Delta_x K(x,y)$

Since $v(x) = \int_{\partial B(0,r)} K(x,y)g(y)dS$, smoothness of K implies $v \in C^\infty(B_r(0))$. To prove claim 2, we have $-\Delta_x v(x) = \int_{\partial B(0,r)} \Delta_x K(x,y)g(y)dS = 0$.

To prove continuity up to the boundary, we first claim

$$\int_{\partial B_r(0)} K(x,y)dS(y) = 1 \ \forall x \in B_r(0) \qquad (\star)$$

Let $w(x) := \int_{\partial B_r(0)} K(x,y)dS(y)$, w satisfies

$$\begin{aligned} -\Delta w = 0, &\quad B_r(0) \\ w = 1 &\quad \partial B_r(0) \end{aligned}$$

$$K(x,y) = \frac{r^2 - |x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n} \qquad w(x) = \int_{\partial B_r(0)} \frac{r^2 - |x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n} \cdot 1 dS$$

by the maximum principle for harmonic function, $w(x) \equiv 1$

so assuming ($\star$),

$$|u(x) - g(x_0)| = |\int_{\partial B_r(0)} K(x,y)(g(y) - g(x_0))dS(y)|$$

$$\leq |\int_{\partial B_r(0)\cap B_\delta(x_0)} K(x,y)(g(y) - g(x_0))dS(y)| + |\int_{\partial B_r(0)\setminus B_\delta(x_0)} K(x,y)(g(y) - g(x_0))dS(y)|$$

since g is continuous on $\partial B_r(0)$, it is uniformly continuous, i.e. there is some $\delta > 0$ such that

$$|\int_{\partial B_r(0)\cap B_\delta(x_0)} K(x,y)(g(y) - g(x_0))dS(y)| \leq \epsilon$$

for the other term, let $|x - x_0| \leq \frac{\delta}{2}$ we have

$$|x_0 - y| > \delta, \ \ |y - x_0| \leq |y - x| + |x - x_0|$$

$$\implies |y - x_0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x_0|$$

$$\implies \frac{1}{2}|y - x_0| \leq |y - x| \implies \frac{1}{|y - x|} \leq \frac{2}{|y - x_0|}$$

we let $C := 2\|g\|_{L^\infty(\partial B(0,r))} \geq |g(y) - g(x_0)|$, then we have

$$|\int_{\partial B_r(0)\setminus B_\delta(x_0)} K(x,y)(g(y) - g(x_0))dS(y)|$$

$$\leq C|\int_{\partial B_r(0)\setminus B_\delta(x_0)} \frac{r^2 - |x|^2}{n\alpha(n)r} \frac{2^n}{|y - x_0|^n}|$$

$$\leq C|\int_{\partial B_r(0)\setminus B_\delta(x_0)} \frac{r^2 - |x|^2}{n\alpha(n)r} \frac{2^n}{\delta^n}| \qquad (\star\star)$$

so as $x \to x_0$, $(r^2 - |x|^2) \to 0$, so $(\star\star) \to 0$, so $|u(x) - g(x)| \to 0$ as $x \to x_0$

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Remark. we can also do a representation formula for

$$\begin{cases} -\Delta u = f \text{ in } B_r(0) \\ u = g \text{ on } \partial B_r(0) \end{cases}$$

energy method (dirichlet principle)

consider the dirichlet problem

$$\begin{cases} -\Delta u = f \text{ in } U & U \text{ open, bounded} \\ u = g \text{ on } \partial U & \partial U \in C^1 \end{cases}$$

we take the associated energy functional

$$I[w] := \int_U \frac{1}{2}|Dw|^2 - wf dx$$

let $\mathcal{A} := \{w \in C^2(\overline{U}) : w = g \text{ on } \partial U\}$

Theorem 0.2. *Dirichlet's principle*
$u \in C^2(\overline{U})$ solves the above dirichlet problem iff $u \in \mathcal{A}$ satisfies

$$I[u] = \min_{w \in \mathcal{A}} I[w]$$

Proof. ($\Longrightarrow$) :
clearly if u satisfies the dirichlet problem, then $u \in \mathcal{A}$. let $w \in \mathcal{A}$, then

$$\int_U (-\Delta u - f)(u - w)dx = 0$$

$$= \int_U Du \cdot D(u - w) - f(u - w)dx - \int_{\partial U} Du \cdot \nu(u - w)dS = 0$$

the second term is equal to 0 because $u = g = w$ on ∂U, so

$$\int_U (Du \cdot Du - Du \cdot Dw - fu + fw)\, dx = 0$$

$$\Longrightarrow \int_U (|Du|^2 - uf)dx = \int_U (Du \cdot Dw - fw)dx$$

because $2ab \leq a^2 + b^2$, we also have

$$Du \cdot Dw \leq |Du| \cdot |Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2$$

$$\Longrightarrow \int_U (|Du|^2 - uf)dx \leq \int_U (\frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2 - wf)dx$$

$$\Longrightarrow I[u] \leq I[w] \; \forall w \in \mathcal{A}$$

($\Longleftarrow$) :
suppose we have $u \in \mathcal{A}$ such that $I[u] \leq I[w] \; \forall w \in \mathcal{A}$
let $v \in C_C^\infty(U)$ and let $i(\tau) := I[u + \tau v]$, $\tau \in \mathbb{R}$
because v is compactly supported in U we will have $u + \tau v \in \mathcal{A}$ for all τ, so $i(\tau)$ reaches a minimum when $\tau = 0$, so $i'(0) = 0$ assuming i is differentiable at 0, so

$$i(\tau) = \int_U \frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v)f dx$$

$$= \int_U \frac{1}{2}|Du|^2 + Du \cdot \tau Dv + \frac{1}{2}\tau^2|Dv|^2 - (u + \tau v)f dx$$

$$\Longrightarrow i'(t) = \int_U (\tau|Dv|^2 + Du \cdot Dv - vf)dx$$

$$\Longrightarrow i'(0) = \int_U Du \cdot Dv - vf = 0$$

$$= \int_U -\Delta uv - vf dx + \int_{\partial U} Du \cdot \nu v dS = 0$$

the second term will equal 0 because $v \equiv 0$ on ∂U, so

$$\int_U (-\Delta u - f) v dx = 0$$

because this holds for every $v \in C_C^\infty(U)$, this implies $-\Delta u = f$ everywhere in U.

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Remark. we can prove the uniqueness of the dirichlet principle using the energy functional

$$\begin{cases} -\Delta w = 0 \text{ in } U \\ w = 0 \text{ on } \partial U \end{cases}$$

$$\implies w \equiv 0$$

the heat equation

let $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$, we study the equation

$$u_t - \Delta u = 0$$

physical motivation: derived from studying the rate of change (in time) of a density

let $u(x, t)$ be equal to the density of some quantity (heat, ink, chemical concentration) at position x and time t

for any $V \subset U$ smooth subdomain, "the rate of change in time is the negation of the net flux through ∂V "

$$\frac{d}{dt} \int_V u(x, t) dx = - \int_{\partial V} F \cdot \nu dS = - \int_V \text{div} F dx$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is the flux density

because V is arbitrary, this implies $u_t = -\text{div} F$ pointwise.

as in laplaces equation, $F = -aDu$, so $u_t - a\Delta u = 0$

Remark. solutions of laplaces equation can be seen as the steady states of solutions of the heat equation, where a steady state is $\lim_{t\to\infty} u(x, t) = v(x)$ so $u_t \to 0$ as $t \to \infty$ implies $-\Delta v = 0$

this implies the properties we saw for harmonic functions should be true (or maybe more complicated) for the heat equation.

Remark. the heat equation is critical to the study of diffusion processes (and probability in general).

the fundamental solution

we want to solve the global cauchy problem

$$\begin{cases} u_t - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x) \text{ in } \mathbb{R}^n \end{cases}$$

just like before, we want to construct a fundamental solution $\Phi(x,t)$ such that

$$\begin{cases} \Phi_t - \Delta Phi = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ \Phi(x,0) = \delta(x) \text{ in } \mathbb{R}^n \end{cases}$$

and we should see

$$u(x,t) := (\Phi(\cdot,t) \star g)(x)$$

solves the GCP

to identify Φ, we seek symmetries of the PDE. we first note if $u(x,t)$ solves $u_t - \Delta u = 0$, then

$$v(x,t) := u(\lambda x, \lambda^2 t)$$

solves

$$v_t - \Delta v = \lambda^2 u_t - \lambda^2 \Delta u = \lambda^2 (u_t - \Delta u) = 0$$

, we call this transformation $(x,t) \mapsto (\lambda x, \lambda^2 t)$ parabolic scaling.

solutions of the heat equation are invariant under parabolic scaling and rotation in x, so

$$\Phi(x,t) = w(\frac{|x|}{t^{\frac{1}{2}}})$$

for some $w : \mathbb{R} \to \mathbb{R}$

we will in fact look for solutions invariant under dilation scaling $u(x,t) \to \lambda^\beta u(\lambda^{\frac{1}{2}} x, \lambda t)$

let $\lambda = \frac{1}{t}$, then we get

$$\frac{1}{t^\beta} u(xt^{\frac{1}{2}}, 1) = t^{-\beta} w(|x| t^{-\frac{1}{2}})$$

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last time $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$

$$\begin{cases} u_t - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \text{ in } \mathbb{R}^n \end{cases}$$

trying to construct a fundamental solution

$$v(x,t) = t^{-\beta} w(|x| t^{-\frac{1}{2}})$$

where $w : \mathbb{R} \to \mathbb{R}$, we have

$$v_t = -\beta t^{-\beta-1} w(|x|t^{-\frac{1}{2}}) + t^{-\beta} w'(|x|t^{-\frac{1}{2}})[-\frac{1}{2} t^{-\frac{3}{2}} |x|]$$

$$v_{x_i} = t^{-\beta} w'(|x|t^{-\frac{1}{2}}) \frac{x_i}{|x|} t^{-\frac{1}{2}}$$

$$v_{x_i x_i} = t^{-\beta-\frac{1}{2}} \Big[w''(|x|t^{-\frac{1}{2}}) \frac{x_i^2}{|x|^2} t^{-\frac{1}{2}} + w'(|x|t^{-\frac{1}{2}}) [\frac{1}{|x|} - \frac{x_i^2}{|x|^3}] \Big]$$

let $r = |x|t^{-\frac{1}{2}}$, then

$$v_t - \Delta v = -\beta t^{-\beta-1}w(r) - \frac{t^{\beta-1}}{2}w'(r)r - \Big[t^{-\beta-1}w''(r) + t^{-\beta-1}\frac{n-1}{|x|t^{-\frac{1}{2}}}w'(r)\Big]$$

$$= t^{-\beta-1}\Big[-\beta w(r) - \frac{1}{2}w'(r)r - w''(r) - \frac{n-1}{r}w'(r)\Big] = 0$$

$$\implies w''(r) + \frac{1}{2}w'(r)r + \frac{n-1}{r}w'(r) + \beta w(r) = 0$$

now, $\beta = \frac{n}{2}$

$$w''(r) + \frac{n-1}{r}w'(r) + \frac{1}{2}\Big[w'(r)r + nw(r)\Big] = 0$$

we multiply by r^{n-1} to get

$$w''(r)r^{n-1} + (n-1)r^{n-2}w'(r) + \frac{1}{2}\Big[w'(r)r^n + nr^{n-1}w(r)\Big]$$

$$= [w'(r)r^{n-1}]' + \frac{1}{2}[w(r)r^n]' = 0$$

$$\implies w'(r)r^{n-1} + \frac{1}{2}w(r)r^n = a$$

if we assume $\lim_{r\to\infty} w(r), w'(r) = 0$, then $a = 0$, so we say

$$w'(r)r^{n-1} + \frac{1}{2}w(r)r^n = 0$$

$$w'(r) = -\frac{1}{2}rw(r)$$

$$\implies w(r) = Ce^{\frac{-|x|^2}{4t}}$$

$$v(x,t) = \frac{1}{t^{\frac{n}{2}}}Ce^{\frac{-r^2}{4}}$$

Definition 0.3. *(fundamental solution of the heat equation)*

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{\frac{-|x|^2}{4t}} \ if\ t > 0 \\ 0\ if\ t < 0 \end{cases}$$

Remark. if $x \neq 0$, then

$$\lim_{t\to 0}\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{\frac{-|x|^2}{4t}} = 0$$

if $x = 0$, then

$$\lim_{t\to 0}\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{\frac{-|x|^2}{4t}} = \infty$$

so $\lim_{t\to 0}\Phi(x,t) \sim \delta(x)$

how did we choose C?

Lemma 0.4.
$$\forall t > 0, \int_{\mathbb{R}^n} \Phi(x,t)dx = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4t}} = 1$$

we now want to solve

$$\begin{cases} u_t - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \text{ in } \mathbb{R}^n \end{cases}$$

we "expect" that

$$u(x,t) := (\Phi(\cdot,t) \star g)(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(y) e^{\frac{-|x-y|^2}{4t}} dy \qquad (\star)$$

Theorem 0.5. *(solution of cauchy problem)*
let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and define u by $(\star)$, then

- $u \in C^\infty(\mathbb{R}^n \times (0,\infty)$
- $u_t - \Delta u = 0$ *in* $\mathbb{R}^n \times (0,\infty)$
- $\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0)$

Proof. since $\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4t}}$ is ∞-differentiable with uniformly bounded (and integrable) derivatives of all orders on $\mathbb{R}^n \times [\delta,\infty)$ $\forall \delta > 0$,

$$u_t(x,t) = \int_{\mathbb{R}^n} \Phi_t(x-y,t) g(y) dy$$

$$\leq ||g|| \int_{\mathbb{R}^n} \Phi_t(x-y,t) dy \leq C(\delta) < \infty$$

similarly, $u_{x_i} \leq C(\delta) < \infty \implies u \in C^\infty(\mathbb{R}^n \times (\delta,\infty))$ $\forall \delta > 0$
$\implies u \in C^\infty(\mathbb{R}^n \times (0,\infty))$
we also have

$$u_t - \Delta u = \int_{\mathbb{R}^n} \Big[(\Phi_t - \Delta\Phi)(x-y,t)\Big] g(y) = 0$$

finally, fix $x_0 \in \mathbb{R}^n$, let $\delta = \delta(x_0) > 0$ s.t. $g(y) - g(x_0) < \epsilon$ whenever $|y - x_0| < \delta$
then as before,

$$u(x,t) - g(x_0)| = |\int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x-y|^2}{4t}} [g(y) - g(x_0)] dy|$$

$$\leq \int_{B(x_0,\delta)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x-y|^2}{4t}} |g(y) - g(x_0)| dy + \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x-y|^2}{4t}} |g(y) - g(x_0)| dy$$

by definition of $\delta > 0$, we have

$$\int_{B(x_0,\delta)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x-y|^2}{4t}} |g(y) - g(x_0)| dy$$

$$\leq \epsilon \int_{B(x_0,\delta)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x-y|^2}{4t}} dy \leq \epsilon$$

also, if $y \in \mathbb{R}^n \setminus B(x_0, \delta)$ but $|x - x_0| \leq \frac{\delta}{2}$, $|y - x_0| > \delta$, then

$$|y - x_0| \leq |y - x| + |x - x_0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{|y - x_0|}{2}$$

so $\frac{|y-x_0|}{2} \leq |y - x|$
thus,

$$\int_{\mathbb{R}^n \setminus B(x_0,\delta)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x-y|^2}{4t}} (g(y) - g(x_0)) dy$$

$$\leq 2 \|g\|_{L^\infty(\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|y-x_0|^2}{16t}} dy$$

$$= C \int_\delta^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-r^2}{16t}} r^{n-1} dr$$

let $r' = \frac{r}{4\sqrt{t}}$, then this is equal to

$$C \int_{\frac{\delta}{4\sqrt{t}}}^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-(r')^2} (4\sqrt{t} r')^{n-1} 4\sqrt{t} dr'$$

$$= C \int_{\frac{\delta}{4\sqrt{t}}}^\infty e^{-(r')^2} (r')^{n-1} dr'$$

which is equal to 0 as $t \to 0$, so as $x \to x_0$, $|x - x_0| \leq \frac{\delta}{2}$ and $t \to 0$, we have $|u(x, t) - g(x_0)| \leq 2\epsilon$

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Remark. if g is bounded, continuous, $g \geq 0$ but $g \neq 0$, then

$$u(x, t) = \int_{\mathbb{R}^n} \underbrace{\frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x-y|^2}{4t}}}_{>0} \underbrace{g(y)}_{\geq 0} dy > 0 \ \forall x \in \mathbb{R}^n \ \forall t \geq 0$$

this means the heat equation has an infinite speed of propagation (the effect of disturbances are felt immediately everywhere).

duhamels formula and source terms

$$\begin{cases} u_t - \Delta u = f(x, t) \text{ in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = 0 \text{ in } \mathbb{R}^n \end{cases}$$

heustic idea

$u_t - \Delta u = f(x,t) \sim \frac{\text{temp}}{\text{time}}$.

suppose we begin at temperature $u = 0$, let $\Delta s > 0$, we begin at time $t = -\Delta s$. we turn on the heat source in the time interval $(-\Delta s, 0)$ and shut it off. at time $t = 0$, the temp $\approx f(x,0)\Delta s$ (if f is continuous).

after time 0, we let the heat equation run, then at time t, the time should be like $w(x,t)\Delta s$ where w solves

$$\begin{cases} w_t - \Delta w = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ w(x,0) = f(x,0) \text{ in } \mathbb{R}^n \end{cases}$$

more generally, if we turn on the heat source $(s-\Delta s, s)$, you get at a later time $t > s$, the temp should be $w(x,t;s)\Delta s$ where $w(\cdot,\cdot;s)$ solves

$$\begin{cases} w_t - \Delta w = 0 \text{ in } \mathbb{R}^n \times (s,\infty) \\ w(x,s) = f(x,s) \text{ in } \mathbb{R}^n \end{cases}$$

so,

$$\begin{cases} u_t - \Delta u = f(x,t) = \sum_{0<\Delta s<\cdots<t} f(x,s)\mathbf{1}_{[s-\Delta s,s]} \\ u(x,0) = 0 \end{cases}$$

by linearity,

$$u(x,t) = \sum_{0<\Delta s<\cdots<t} w(x,t;s)\Delta s \approx \int_0^t w(x,t;s)ds$$

remember, $w(x,t;s)$ solves heat equation in $\mathbb{R}^n \times (s,\infty)$

$$\implies w(x,t;s) = v(x,t-s;s)$$

where v solves

$$\begin{cases} v_t - \Delta v = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ v(x,0;s) = f(x,s) \text{ in } \mathbb{R}^n \end{cases}$$

$$v(x,t;s) = \Big[\Phi(\cdot,t) \star f(\cdot,s)\Big](x)$$

$$v(x,t-s;s) = \Big[\Phi(\cdot,t-s) \star f(\cdot,s)\Big](x)$$

$$\begin{aligned} u(x,t) &:= \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{\frac{-|x-y|^2}{4(t-s)}} f(y,s)dyds \qquad (\Delta) \\ &= \int_0^t v(x,t-s;s)ds \end{aligned}$$

Theorem 0.6. *(solutions of a nonhomogenous problem)*
let u be defined by Δ, *where* $f \in C^{2,1}(\mathbb{R}^n \times [0,\infty))$ *and* f *has compact support in space & for large t, then*

- $u \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$
- $u_t - \Delta u = f(x,t) \in \mathbb{R}^n \times (0,\infty)$
- $\lim_{(x,t)\to(x_0,0)} u(x,t) = 0 \,\forall x_0 \in \mathbb{R}^n$

Proof. note $\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4t}}$ is singular at $(0,0)$.

$$\Phi_t \sim C t^{-\frac{n}{2}-1} e^{\frac{-|x|^2}{4t}} + C_1 t^{-\frac{1}{2}n} e^{\frac{-|x|^2}{4t}} (-\frac{|x|^2}{t^2}) \sim t^{-\frac{n}{2}} t^{-2}$$

$$\int_0^t \int_{\mathbb{R}^n} \Phi_t(x-y, t-s) f(y,s) dyds$$
$$\sim \int_0^t \int_{\mathbb{R}^n} C(t-s)^{n-2}(t-s)^{-2} e^{\frac{-|x-y|^2}{4(t-s)}} f(y,s) dyds$$
$$\sim \int_0^t C(s)(t-s)^{-2} ds$$

which is not integrable at $s = t$

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instead, we consider

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s) dyds$$

we notice $\Phi(y,s)\big|_{s=t}$ is smooth

$$\implies u_t = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y,t-s) dyds + \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,0) dy < \infty$$

similarly,

$$u_{x_i x_i} = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_{x_i x_i}(x-y,t-s) dyds < \infty$$

Remark. f is compactly supported in space (do not need $f(x,0) = 0$)

$$\implies u_t, D^2 u \in C(\mathbb{R}^n \times (0,\infty)) \implies u \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$$

then

$$u_t - \Delta u = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\Big[(\frac{\partial}{\partial t} - \Delta_x) f(x-y,t-s)\Big] dyds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0) dy$$

$$= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\Big[(-\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)\Big] dyds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0) dy$$

$$= \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s)\Big[\cdot\Big] dyds + \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\Big[\cdot\Big] dyds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0) dy$$

so

$$\int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s)\Big[(-\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)\Big] dyds$$

$$\leq (\|f_t\|_{L^\infty} + \left\|D^2 f\right\|_{L^\infty}) \int_0^\epsilon \underbrace{\int_{\mathbb{R}^n} \Phi(y,s) dy}_{=1} ds$$

$$= C\epsilon \to 0$$

$$\int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\Big[(-\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)\Big] dyds$$

$$= \int_\epsilon^t \int_{\mathbb{R}^n} \underbrace{(-\frac{\partial}{\partial s} - \Delta_y)\Phi(y,s)}_{=0} f(x-y,t-s) dyds - \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s) dy \Big|_\epsilon^t$$

$$+ \text{ terms for which } f \text{ has compact support, so } \approx 0$$

$$= -\int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0) dy + -\int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon) dy$$

$$\implies u_t - \Delta u = \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon) dy$$

take $\epsilon \to 0$ to get

$$\int_{\mathbb{R}^n} \delta(y) f(x-y,t) dy = f(x,t)$$

as desired.
finally, by duhamel,

$$\|u(\cdot,t)\|_{L^\infty} \leq t\|f\|_{L^\infty} \implies \lim_{t\to 0} \|u(\cdot,t)\|_{L^\infty} = 0$$

■

more generally,

$$\begin{cases} u_t - \Delta u = f \text{ in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g \text{ in } \mathbb{R}^n \end{cases}$$

is solved by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) dyds$$

boundary value problems and some results

let $U \subset \mathbb{R}^n$ open bounded, we call

$$U_T := U \times (0, T]$$

a parabolic cylinder

we call

$$\partial_p U_T := \partial U \times [0, T] \cup U \times \{t = 0\}$$

the parabolic boundary.

$$\overline{U_T} = U_T \cup \partial_p U_T$$

we are interested in solving the dirichlet problem

$$\begin{cases} u_t - \Delta u = 0 \text{ in } U_T \\ u = g \text{ on } \partial_p U_T \end{cases}$$

or the neumann problem

$$\begin{cases} u_t - \Delta u = 0 \text{ in } U_T \\ u(x, 0) = g(x) \text{ in } U \times \{t = 0\} \\ \partial_\nu u(x, t) = h(x, t) \text{ on } \partial U \times [0, T] \end{cases}$$

Remark. there IS a mean value property for the heat equation, there are space time domains

$$E(x, t; r) := \{(y, s) \in \mathbb{R}^{n+1} : s \leq t, \Phi(x - y, t - s) \geq \frac{1}{r^n}\} \subset U_T$$

"heat ball"

$$u(x, t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y, s) \frac{|x - y|^2}{(t - s)^2} dyds$$

consequences

Theorem 0.7. *(regularity)*
$u \in C^{2,1}(U_T)$ solves the heat equation in $U_T \implies u \in C^\infty(U_T)$

let $Q_r(x, t) := B(x, t) \times (t - r^2, t]$

Theorem 0.8. *(derivative bounds) if u solves the heat equation in U_T, then*

$$\forall Q_r(x, t) \subset U_T \quad \max_{Q_{\frac{r}{2}}(x,t)} |D_x^k D_t^\ell u \leq \frac{C_{kl}}{r^{k+2\ell+n+2}} \|u\|_{L^1(Q_r(x,t))}$$

we also get the same comparison/max principle results.

Theorem 0.9. *(harnack inequality) let u be a nonnegative solution of the heat equation in U_T. let $K \subset\subset U$, let $\tau \in (0, T)$. there exists $C = C(K, \tau, t - \tau)$ s.t. $\forall t \in (\tau, T)$,*

$$\sup_K u(\cdot, t - \tau) \le C \inf_K u(\cdot, t)$$

this is an example of causality, the future cannot influence the past.
this also shows the infinite speed of propagation!

$$u(\cdot, t - \tau) > 0 \implies u(\cdot, t) > 0$$

Theorem 0.10. *(strong max principle)*
if U is connected and $\exists (x_0, t_0) \in U_T$ s.t. $u(x_0, t_0) = \max_{\overline{U_T}} u$, then u is constant in $\overline{U_{t_0}}$

Proof. let $w(x, t) := \max_{\overline{U_T}} u - u(x, t) \ge 0$ and $w(x_0, t_0) = 0$
by harnack inequality,

$$0 \le \sup_K w(\cdot, t_0 - \tau) \le 0 \ \ \forall \tau \in (0, t_0)$$

$$\implies w \equiv 0 \in \overline{U_{t_0}} \implies u \text{ constant in } \overline{U_{t_0}}$$

■

Theorem 0.11. *(max and comparison principle)*
let $u \in C^{2,1}(U_T) \cap C(\overline{U_T})$ satisfy $u_t - \Delta u \le 0$ in U_T, then

$$\max_{\overline{U_T}} u = \max_{\partial_p U_T} u$$

this gives

Theorem 0.12. *(uniqueness of DP)*
there is at most one solution $u \in C^{2,1}(U_T) \cap C(\overline{U_T})$ s.t.

$$\begin{cases} u_t - \Delta u = f \text{ in } U_T \\ u = g \text{ on } \partial_p U_T \end{cases}$$

energy methods

$$\begin{cases} u_t - \Delta u = f \text{ in } U_T \\ \bigoplus \text{ B.C.} \end{cases} \tag{$\star$}$$

Theorem 0.13. *(uniqueness) $\exists$ at most 1 solution of $(\star)$ belonging to $C^{2,1}(\overline{U_T})$*

Proof. uniqueness is equivalent to showing

$$\begin{cases} w_t - \Delta w = 0 \text{ in } U_T \\ w(x, 0) = 0 \text{ in } U \\ w \text{ has } 0 \text{ B.C.} \end{cases}$$

$$\implies w \equiv 0$$

let

$$E(t) := \int_U w^2(x,t)dx \quad E(t) \geq 0$$

$$E'(t) = \int_U 2ww_t dx = \int_U 2w\Delta w dx$$

$$= -\int_U 2|\Delta w|^2 dx + \underbrace{\int_{\partial U} 2w\Delta \cdot \nu dS}_{=0} \leq 0$$

so

$$E(0) = \int_U w^2(x,0)dx = 0$$

$$0 \leq E(t) \leq E(0) = 0 \implies \int_U w^2(x,t)dx = 0 \;\; \forall t$$

$$\implies w \equiv 0$$

■

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Theorem 0.14. *(maximum principle for cauchy problem)*
let $u \in C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$
solve

$$\begin{cases} u_t - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,T] \\ u(x,0) = g(x) \text{ in } \mathbb{R}^n \end{cases}$$

and suppose $u(x,t) \leq Ae^{a|x|^2}$ in $\mathbb{R}^n \times [0,T]$ for some $A, a > 0$
then

$$\sup_{\mathbb{R}^n \times [0,t]} u = \sup_{\mathbb{R}^n} g$$

Proof. assume $4aT < 1$

$$\implies \exists \epsilon > 0 \text{ s.t. } 4a(T+\epsilon) < 1 \tag{$\star$}$$

fix $y \in \mathbb{R}^n$, $\mu > 0$, let

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\epsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}}$$

this is like Φ but $t \to T+\epsilon-t$, $e^{-|x|^2} \to e^{+|x|^2}$, this still solves the heat equation, so

$$v_t - \Delta v = 0 \text{ in } \mathbb{R}^n \times (0,T]$$

let $r > 0$, $U = B(y, r)$, $U_T := B(y, r) \times (0, T]$
since $v_t - \Delta v = 0$ in U_T, by the maximum principle $\max_{\overline{U_T}} v = \max_{\partial_p U_T} v$
note

$$v(x, 0) = u(x, 0) - \frac{\mu}{(T+\epsilon)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\epsilon)}} \leq u(x, 0) = g(x) \leq \sup_{\mathbb{R}^n} g$$

if $(x, t) \in \partial_p U_T$, $|x - y| = r$, $t \in (0, T]$

$$v(x, t) = u(x, t) - \frac{\mu}{(T+\epsilon-t)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\epsilon-t)}}$$

$$\leq Ae^{a|x|^2} - \frac{\mu}{(T+\epsilon-t)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\epsilon-t)}}$$

$|x| \leq |y| + |x - y|$

$$\leq Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\epsilon)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\epsilon)}}$$

by $(\star)$, $\frac{1}{4a(T+\epsilon)} > 1$, $\frac{1}{4(T+\epsilon)} = a + \gamma$ for some $\gamma > 0$

$$\leq Ae^{a(|y|+r)^2} - \mu(4(a+\gamma))^{\frac{n}{2}} e^{r^2(a+\gamma)}$$

we can choose r sufficiently large (depending on y) s.t.

$$\leq \sup_{\mathbb{R}^n} g \implies v(x, t) \leq \sup_{\mathbb{R}^n} g \text{ on } \partial_p U_T$$

we can then send $\mu \to 0$ to see $u(y, t) \leq \sup_{\mathbb{R}^n} g$ for $y \in \mathbb{R}^n$, $t \in (0, T]$ as desired.
if $(\star)$ does not hold, we can repeat this argument and apply to time intervals

$$[0, T_1], \quad [T_1, 2T_1], \quad \ldots [kT_1, T]$$

where $4aT_1 < 1$, and using that $u_1(x, t) := u(x, t + T_1)$ solves

$$\begin{cases} (u_1)_t - \Delta(u_1) = 0 \text{ in } \mathbb{R}^n \times (0, T - T_1) \\ u_1(x, 0) = u(x, T_1) \end{cases}$$

■

Theorem 0.15. *(uniqueness for maximum principle)*
let $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0, T])$. there exists at most 1 solution $u \in C^{2,1}(\mathbb{R}^n \times (0, T]) \cap C(\mathbb{R}^n \times [0, T])$ of

$$\begin{cases} u_t - \Delta u = f(x, t) \text{ in } \mathbb{R}^n \times (0, T] \\ u(x, 0) = g(x) \text{ in } \mathbb{R}^n \end{cases}$$

satisfying $|u(x, t)| \leq Ae^{a|x|^2}$

Proof. let u_1, u_2 be two such solutions, apply maximum principle for cauchy problem to $u := u_1 - u_2$, which will solve

$$\begin{cases} u_t - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, T] \\ u(x, 0) = 0 \text{ in } \mathbb{R}^n \end{cases}$$

so $\sup |u| = 0$ ■

Remark. sufficient condition for growth bound is

$$|g(x)| \leq Ce^{\gamma|x|^2} \quad C, \gamma > 0 \tag{$\star$}$$

$$\implies u(x, t) := (\Phi(\cdot, t) \star g)(x)$$

satisfies $|u(x, t)| \leq Ae^{a|x|^2}$

so this is the unique solution, for any initial condition satisfying ($\star$)

Remark. $|u(x, t)| \leq Ae^{a|x|^2}$ is called a tychonoff condition, without it, there is no uniqueness! there are infinitely many solutions to

$$\begin{cases} u_t - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, T] \\ u(x, 0) = 0 \text{ in } \mathbb{R}^n \end{cases}$$

and for the nonzero solutions, $|u(x, t)| \nleq Ae^{a|x|^2}$

the tychonoff condition picks up physically relevant solutions.

the wave equation

$$u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$$

$$u_{tt} - \Delta u := \Box u = 0$$

and

$$u_{tt} - \Delta u = \Box u = f(x, t)$$

physical interpretation

$u(x, t)$ represents displacement of vibrating string/membrane/solid ($n = 1/2/3$ respectively)

newtons law: $F = ma$

$\forall V \subset \mathbb{R}^n, -\int_{\partial V} F \cdot \nu dS = \int_V u_{tt} dx$

$$-\int_V \text{div}(F) dx = \int_V u_{tt} dx \quad \forall V \subset \mathbb{R}^n$$

$$\implies u_{tt} = -\text{div} F$$

if $F = F(Du) \approx -aDu$

$$u_{tt} = a\text{div}(Du) = a\Delta u$$

$$\implies u_{tt} - a\Delta u = 0$$

we will study the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u_(x, 0) = g(x) \text{ in } \mathbb{R}^n \text{ displacement} \\ u_t(x, 0) = h(x) \text{ in } \mathbb{R}^n \text{ velocity} \end{cases}$$

why do we need $u_t(x, 0)$? without it, if u solves the wave equation, then $u + kt$ solves it for all $k \in \mathbb{R}$

dalemberts formula

$n = 1$

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u_(x, 0) = g(x) \text{ in } \mathbb{R} \\ u_t(x, 0) = h(x) \text{ in } \mathbb{R} \end{cases}$$

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})u = u_{tt} - u_{xx} = 0$$

let

$$v(x, t) := (\frac{\partial}{\partial t} - \frac{\partial}{\partial x})u \implies v_t + v_x = 0 \text{ in } \mathbb{R} \times (0, \infty)$$

we note that

$$v(x, t) := a(x - t)$$

satisfies

$$v_t + v_x = -a'(x - t) + a'(x - t) = 0 \text{ with } v(x, 0) = a(x)$$

thus,

$$a(x - t) = u_t - u_x$$

claim:

$$u(x, t) = \frac{1}{2}\int_{x-t}^{x+t} a(y)dy + b(x + t)$$

check:

$$u_t = \frac{1}{2}a(x + t) + \frac{1}{2}a(x - t) + b'(x + t)$$

$$u_x = \frac{1}{2}a(x + t) - \frac{1}{2}a(x - t) + b'(x + t)$$

$$\implies u_t - u_x = a(x - t)$$

so

$$u(x, 0) = b(x) = g(x)$$

$$v(x, 0) = a(x) = u_t(x, 0) - u_x(x, 0) = h(x) - g'(x)$$

$$\implies u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(y) - g'(y)dy + g(x+t)$$

$$= -\frac{1}{2}g(x+t) + \frac{1}{2}g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy + g(x+t)$$

$$u(x,t) = \frac{1}{2}g(x+t) + \frac{1}{2}g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy$$

Theorem 0.16. *(dalemberts formula) assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, u defined as above. then*

- $u \in C^2(\mathbb{R} \times (0,\infty))$
- $u_{tt} - u_{xx} = 0$ *in* $\mathbb{R} \times (0,\infty)$
- $\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0)$
- $\lim_{(x,t)\to(x_0,0)} u_t(x,t) = h(x_0)$

Proof. clearly, $u \in C^2(\mathbb{R} \times (0,\infty))$

$$u(x,t) = \frac{1}{2}g(x+t) + \frac{1}{2}g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy$$

$$u_t = \frac{1}{2}g'(x+t) - \frac{1}{2}g'(x-t) + \frac{1}{2}h(x+t) + \frac{1}{2}h(x-t)$$

$$u_{tt} = \frac{1}{2}g''(x+t) + \frac{1}{2}g''(x-t) + \frac{1}{2}h'(x+t) - \frac{1}{2}h'(x-t)$$

$$u_{xx} = \frac{1}{2}g''(x+t) + \frac{1}{2}g''(x-t) + \frac{1}{2}h'(x+t) - \frac{1}{2}h'(x-t)$$

$$\implies u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R} \times (0,\infty)$$

$$\lim_{(x,t)\to(x_0,0)} u_t(x,t)$$

$$= \lim_{(x,t)\to(x_0,0)} \frac{1}{2}g'(x+t) - \frac{1}{2}g'(x-t) + \frac{1}{2}h(x+t) + \frac{1}{2}h(x-t)$$

$$= h(x_0)$$

$$\lim_{(x,t)\to(x_0,0)} u(x,t)$$

$$= \lim_{(x,t)\to(x_0,0)} \frac{1}{2}g(x+t) + \frac{1}{2}g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy$$

$$= g(x_0)$$

■

Remark. $g \in C^k, h \in C^{k-1} \implies u \in C^k$, nothing better! no regularizing effect!!

Remark. this in fact is the unique solution! this is a consequence of the factoring and first order uniqueness theory

$$u(x,t) = F(x+t) + G(x-t)$$

and this is enough to deduce the dalembert formula.

Remark. using the dalembert formula, you can get "max principle" and stability estimates.

$$|u(x,t) - v(x,t)| \leq \max|u(x,0) - v(x,0)| + 2t \max|u_t(x,0) - v_t(x,0)|$$

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reflection method

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}_+ \times (0,\infty) \\ u_(x,0) = g(x) \text{ in } \mathbb{R}_+ \\ u_t(x,0) = h(x) \text{ in } \mathbb{R}_+ \\ u(0,t) = 0 \\ g(0) = h(0) = 0 \end{cases}$$

how do we extend to all of $\mathbb{R}$ to use dalembert so that $u(0,t) = 0$? we seek to build a solution $\tilde{u}(x,t)$ defined on all of $\mathbb{R} \times (0,\infty)$, with $\tilde{u}(\cdot,t)$ odd.

to ensure $\tilde{u}(0,t)$ is odd, we use odd reflection.

$$\tilde{g}(x) := \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x < 0 \end{cases}$$

$$\tilde{h}(x) := \begin{cases} h(x) & x \geq 0 \\ -h(-x) & x < 0 \end{cases}$$

$$\implies \begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 \text{ in } \mathbb{R} \times (0,\infty) \\ \tilde{u}_(x,0) = \tilde{g}(x) \text{ in } \mathbb{R} \\ \tilde{u}_t(x,0) = \tilde{h}(x) \text{ in } \mathbb{R} \end{cases}$$

by dalembert,

$$\tilde{u}(x,t) = \frac{1}{2}\tilde{g}(x+t) + \frac{1}{2}\tilde{g}(x-t) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)dy$$

and

$$\tilde{u}(-x,t) = \frac{1}{2}\tilde{g}(-x+t) + \frac{1}{2}\tilde{g}(-x-t) + \frac{1}{2}\int_{-x-t}^{-x+t} \tilde{h}(y)dy$$

$$= -\frac{1}{2}\tilde{g}(x-t) - \frac{1}{2}\tilde{g}(x+t) - \frac{1}{2}\int_{x+t}^{x-t} \tilde{h}(-y)dy$$

$$= -\frac{1}{2}\tilde{g}(x-t) - \frac{1}{2}\tilde{g}(x+t) - \frac{1}{2}\int_{x-t}^{x+t}\tilde{h}(y)dy$$

$$= -\tilde{u}(x,t) \implies u(\cdot,t) \text{ is odd}$$

so now letting $u(x,t) := \tilde{u}(x,t)$ in $\mathbb{R}_+ \times (0,\infty)$,

$$u(x,t) = \begin{cases} \frac{1}{2}g(x+t) + \frac{1}{2}g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy & \text{if } x \geq t \\ \frac{1}{2}g(x+t) - \frac{1}{2}g(t-x) + \frac{1}{2}\int_{t-x}^{x+t} h(y)dy & \text{if } x < t \end{cases}$$

Remark. we need $g''(0) = 0$ to make this C^2

solutions of wave equation in higher dimensions

$$(CP)\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ u_(x,0) = g(x) \text{ in } \mathbb{R}^n \\ u_t(x,0) = h(x) \text{ in } \mathbb{R}^n \end{cases}$$

$$u \in C^m(\mathbb{R}^n \times (0,\infty)) \;\; m \geq 2$$

idea: we want to identity a quantity which solves a 1D wave equation. that quantity is going to be averages of u on spherical shells.

in odd dimensions, this works, in even dimensions, we project down.

let $x \in \mathbb{R}^n, t > 0, r > 0$

$$U(r,t;x) := \fint_{\partial B(x,r)} u(y,t)dS(y)$$

$$G(r;x) := \fint_{\partial B(x,r)} g(y)dS(y)$$

$$H(r;x) := \fint_{\partial B(x,r)} h(y)dS(y)$$

Lemma 0.17. *(euler-poisson-darboux equation)*

if $u \in C^m$ solves (CP), then $U \in C^m\mathbb{R} \times (0,\infty))$ and

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0 \\ U(r,0;x) = G(r,x) \\ U_t(r,0;x) = H(r;x) \end{cases}$$

Proof. recall MVP prood

$$U_r(r,t;x) = \frac{\partial}{\partial r}\fint_{\partial B(x,r)} u(y,t)dS(y)$$

$$= \frac{1}{|\partial B(x,r)|}\int_{B(x,r)} \Delta u(y,t)dy$$

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y,t)dy$$

$$= \frac{r}{n} ⨍_{B(x,r)} \Delta u(y,t)dy$$

$u \in C^m$, $m \geq 2$, U_r is definted in $\mathbb{R}_+ \times (0,\infty)$ and $\lim_{r\to 0} U_r(r,t;x) = 0$
similarly,

$$U_{rr}(r,t;x) = \frac{\partial}{\partial r}\Big(\frac{1}{n\alpha(n)} r^{1-n} \int_{B(x,r)} \Delta u(y,t)dy\Big) \qquad (\star)$$

Remark.

$$\int_{B(x,r)} f(y)dy = \int_0^r \int_{\partial B(x,\sigma)} f(y)dS\, d\sigma$$

$$\implies \frac{\partial}{\partial r}\Big(\int_{B(x,r)} f(y)dy\Big) = \int_{\partial B(x,r)} f(y)dS$$

$$(\star) = \frac{1}{\alpha n}\Big[r^{1-n} \int_{\partial B(x,r)} \Delta u(y,t)dS(y) + (1-n)r^{-n} \int_{B(x,r)} \Delta u(y,t)dy\Big]$$

$$= ⨍_{\partial B(x,r)} \Delta u(y,t)dS(y) + \frac{1-n}{n} ⨍_{B(x,r)} \Delta u(y,t)dy$$

$$\lim_{r\to 0} U_{rr}(r,t;x) = \Delta u(x,t) + (\frac{1}{n} - 1)\Delta u(x,t)$$

$$= \frac{1}{n}\Delta u(x,t) < \infty$$

similarly, U_{rrr}
so from above,

$$U_r = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y,t)dy$$

$$= \frac{r}{n} ⨍_{B(x,r)} u_{tt}(y,t)dy$$

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} u_{tt}dy$$

$$\implies r^{n-1}U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}dy$$

note,

$$r^{n-1}U_{rr} + (n-1)r^{n-2}U_r = (r^{n-1}U_r)_r$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}dS(y) = r^{n-1} ⨍_{\partial B(x,r)} u_{tt}dS = r^{n-1}U_{tt}$$

$$\implies U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0$$

■

solution of wave equation, n = 3

suppose $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ solves

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^3 \times (0,\infty) \\ u(x,0) = g \text{ in } \mathbb{R}^3 \\ u_t(x,0) = h \text{ in } \mathbb{R}^3 \end{cases}$$

let $U(r,t;x), G(r;x), H(r;x)$ as before

$$\tilde{U}(r,t;x) := rU(r,t;x)$$
$$\tilde{G}(r;x) := rG(r;x)$$
$$\tilde{H}(r;x) := rU(r;x)$$

$$\implies \tilde{U}_{tt} = rU_{tt}$$
$$\tilde{U}_r = rU_r + U$$
$$\tilde{U}_{rr} = rU_{rr} + U_r + U_r = rU_{rr} + 2U_r$$

$$\tilde{U}_{tt} = rU_{tt} = r[U_{rr} + \frac{2}{r}U_r]$$
$$= rU_{rr} + 2U_r = \tilde{U}_{rr}$$

$$\implies \begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 \text{ in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U}(r,0;x) = \tilde{G}(r;x) \text{ in } \mathbb{R}_+ \\ \tilde{U}_t(r,0;x) = \tilde{H}(r;x) \text{ in } \mathbb{R}_+ \\ \tilde{U}(0,t;x) = 0 \text{ in } (0,\infty) \end{cases}$$

Remark. $g \in C^m, \tilde{G}_r = G + rG_r$

$$\tilde{G}_{rr} = rG_{rr} + 2G_r$$
$$\implies \lim_{r\to 0} \tilde{G}_{rr} = \lim_{r\to 0} 2G_r$$
$$= \lim_{r\to 0} \frac{2r}{n} \fint_{B(x,r)} \Delta g(y)dy = 0$$
$$\implies \tilde{G}_{rr}(0,t;x) = 0$$

$\implies$ by reflection and dalembert, if $0 \le r \le t$,

$$\tilde{U}(r,t;x) := \frac{1}{2}\Big[\tilde{G}(r+t) - \tilde{G}(t-r)\Big] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)dy$$

$$u \in C^m$$

also, by continuity,

$$\begin{aligned} u(x,t) &= \lim_{r\to 0} \fint_{\partial B(x,r)} u(y,t)dS \\ &= \lim_{r\to 0} U(r,t;x) \\ &= \lim_{r\to 0} \frac{\tilde{U}(r,t;x)}{r} \end{aligned}$$

$$\implies u(x,t) = \lim_{r\to 0} \frac{1}{2r}\Big[\tilde{G}(t+r) - \tilde{G}(t-r)\Big] + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)dy$$

$$= \tilde{G}'(t) + \tilde{H}(t)$$

$$u(x,t) = \frac{\partial}{\partial t}\Big[t \fint_{\partial B(x,t)} g(y)dS(y)\Big] + t\fint_{\partial B(x,t)} h(y)dS(y)$$

kirchoffs formula of (CP) in $\mathbb{R}^3 \times (0,\infty)$

Remark.
- only need g, h defined on shells
- regularity is dependant on g, h, $\quad g \in C^{m+1} \implies u \in C^m$

wave equation in n=2

we use the formula from $n = 3$ and the method of descent
idea: embed a 2D problem in 3D

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^2 \times (0,\infty) \\ u(x,0) = g \text{ in } \mathbb{R}^2 \\ u_t(x,0) = h \text{ in } \mathbb{R}^2 \end{cases}$$

let $\underbrace{\overline{u}(x_1,x_2,x_3,t)}_{\text{constant in } \mathbb{R}^3} := u(x_1,x_2,t), \overline{u} : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}$

$$\overline{u}(x_1,x_2,x_3,0) := g(x_1,x_2) = \overline{g}(x_1,x_2,x_3)$$

$$\overline{u}_t(x_1,x_2,x_3,0) := h(x_1,x_2) = \overline{h}(x_1,x_2,x_3)$$

then we still have

$$\begin{cases} \overline{u}_{tt} - \Delta\overline{u} = 0 \text{ in } \mathbb{R}^3 \times (0,\infty) \\ \overline{u}(x,0) = \overline{g} \text{ in } \mathbb{R}^3 \\ \overline{u}_t(x,0) = \overline{h} \text{ in } \mathbb{R}^3 \end{cases}$$

$$\implies \overline{u}(x_1,x_2,x_3,t) = \frac{\partial}{\partial t}\Big[t\fint_{\partial\overline{B}(\overline{x},t)} \overline{g}d\overline{S}(y)\Big] + t\fint_{\partial\overline{B}(\overline{x},t)} \overline{h}d\overline{S}(y)$$

$\overline{B}(\overline{x},t)$ is ball in $\mathbb{R}^3$ centered at $\overline{x}, t$
$d\overline{S}$ = surface measure in $\mathbb{R}^3$
observe, $\partial\overline{B}(\overline{x},t) \subset \mathbb{R}^3$ is the union of 2 hemispheres,

$$y_3 = \gamma \pm (y_1, y_2) = x_3 \pm \sqrt{t^2 - |y-x|^2}$$

$$(\gamma_\pm) = \pm\frac{1}{2}(t^2 - |y-x|^2)^{-\frac{1}{2}} 2(y_i - x_i)$$

on each hemisphere,

$$d\overline{S}(y) = \sqrt{1 + (D_\gamma)^2} dy$$

$$= \sqrt{1 + \frac{(x_1 - y_1)^2}{t^2 - |y-x|^2} + \frac{(x_2 - y_2)^2}{t^2 - |y-x|^2}} dy$$

$$= \sqrt{1 + \frac{|x-y|^2}{t^2 - |x-y|^2}} dy$$

$$= \frac{t}{\sqrt{t^2 - |x-y|^2}} dy$$

$$\overline{u}(x_1, x_2, x_3, t) = \frac{\partial}{\partial t}\Big[t \fint_{\partial\overline{B}(\overline{x},t)} \overline{g} d\overline{S}(y)\Big] + t \fint_{\partial\overline{B}(\overline{x},t)} \overline{h} d\overline{S}(y)$$

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$$\overline{u}(x_1, x_2, x_3, t) = \frac{\partial}{\partial t}\Big[t \frac{1}{4\pi t^2} 2 \int_{B(x,t)} \frac{g(y)t}{\sqrt{t^2 - |x-y|^2}} dy\Big] + t \frac{2}{4\pi t} \int_{B(x,t)} \frac{h(y)t}{\sqrt{t^2 - |x-y|^2}} dy$$

note: RHS is independent of x_3!!
note: $B(x,t)$ refers to the ball in $\mathbb{R}^2$

$$= \frac{\partial}{\partial t}\Big(\frac{t^2}{2} \fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}} dy\Big) + \frac{t^2}{2} \fint_{B(x,t)} \frac{h(y)}{\sqrt{t^2 - |x-y|^2}} dy$$

this is poissons formula for the wave equation in $\mathbb{R}^2 \times (0, \infty)$

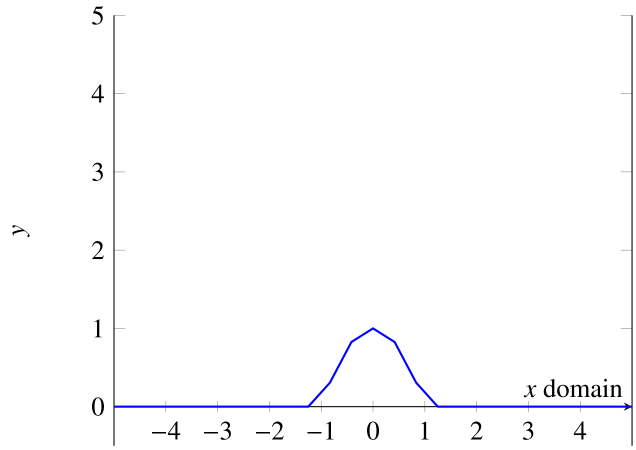
Remark. still need $g \in C^{m+1}, h \in C^m \implies u \in C^m$

propagation speed, domain of dependance, region of influence

take $n = 1$

$$u(x,t) = \frac{1}{2} g(x+t) + \frac{1}{2} g(x-t) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy$$

if g has compact support, $h = 0$, the graph of g will propagate to the left and right as time passes at a constant speed.



if $g, h \neq 0$, the graph will propagate taking values from a larger subset of the support of h as t grows.

we say the wave equation has a finite speed of propagation because disturbances propagate with finite speed (1)

Remark. $u_{tt} - c^2 \Delta u = 0$ implies propatation speed is c

Definition 0.18. *(domain of dependance) cone for each* $(x_0, t_0) \in \mathbb{R} \times (0, \infty)$*, places which can influence* (x_0, t_0)

$n = 3$

$$u(x,t) = \frac{\partial}{\partial t}\Big[t \fint_{\partial B(x,t)} g(y) dS(y)\Big] + t \fint_{\partial B(x,t)} h(y) dS(y)$$

need g, h on $\partial B(x,t)$
region of influence

$$\{(x,t) : \partial B(x,t) \cap (\mathrm{supp}(g) \cup \mathrm{supp}(h)) \neq \emptyset\}$$

domain of dependance of (x_0, t_0) $\partial B(x_0, t_0) \times \{t = 0\}$
$n = 2$

$$u(x,t) = \frac{\partial}{\partial t}\Big(\frac{t^2}{2} \fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}} dy\Big) + \frac{t^2}{2} \fint_{B(x,t)} \frac{h(y)}{\sqrt{t^2 - |x-y|^2}} dy$$

region of influence: $\{(x,t) : B(x,t) \cap (\mathrm{supp}(g) \cup \mathrm{supp}(h)) \neq \emptyset\}$
domain of dependence: $B(x_0, t_0) \times \{t = 0\}$
we can prove this via energy method

Theorem 0.19. *(finite propagation speed) if* $u \equiv u_t \equiv 0$ *in* $B(x_0, t_0) \times \{t = 0\}$

$$\implies u \equiv 0 \text{ in } C = \text{ cone}$$

Proof. (via energy) let

$$e(t) := \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2 + |Du|^2 dx$$

$e(t) \geq 0$

$$e(0) = \frac{1}{2}\int_{B(x_0,t_0)} \underbrace{u_t^2}_{=0} + \underbrace{|Du|^2}_{=0} dx = 0$$

$$e'(t) = \frac{1}{2}\int_{B(x_0,t_0-t)} 2u_t u_{tt} + 2Du \cdot Du_t dx - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2 + |Du|^2 dS$$

(integration by parts)

$$= \underbrace{\int_{B(x_0,t_0-t)} u_t u_{tt} - \Delta u u_t dx}_{=0 \text{ by wave equation}} + \int_{\partial B(x_0,t_0-t)} u_t Du \cdot \nu - \frac{1}{2}u_t^2 - \frac{1}{2}|Du|^2 dS$$

$$= \int_{\partial B(x_0,t_0-t)} u_t \frac{\partial u}{\partial \nu} - \frac{1}{2}u_t^2 - \frac{1}{2}|Du|^2 dS \qquad (\star)$$

$|u_t \frac{\partial u}{\partial \nu}| \leq |u_t||Du| \leq \frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2$

$$\implies (\star) \leq 0$$

$$\implies 0 \leq e(t) \leq e(0) = 0 \quad \forall 0 \leq t \leq t_0$$

$$0 = \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2 + |Du|^2 dx$$

$$\implies u_t \equiv Du \equiv 0$$

and $u(x,0) \equiv 0$ in $B(x_0,t_0)$

$$\implies u \equiv 0 \text{ in } C$$

$$C = \bigcup_{t=0}^{t_0} B(x_0, t_0 - t)$$

■

duhamel for the wave equation

$$\begin{cases} u_{tt} - \Delta u = f(x,t) \text{ in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = 0 \text{ in } \mathbb{R}^n \\ u_t(x,0) = 0 \text{ in } \mathbb{R}^n \end{cases}$$

same analysis as before,

$$u(x,t) = \int_0^t v(x, t-s; s) ds \qquad (\star)$$

where

$$\begin{cases} v_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ v(x, 0) = 0 \text{ in } \mathbb{R}^n \\ v_t(x, 0) = f(x, s) \text{ in } \mathbb{R}^n \end{cases}$$

Theorem 0.20. *assume $n \geq 2$, $f \in C^2$, let u defined by $(\star)$, then*

- $u \in C^2(\mathbb{R}^n \times (0, \infty))$
- $u_{tt} - \Delta u = f(x, t)$ *in* $\mathbb{R}^n \times (0, \infty)$
- $\lim_{(x,t) \to (x_0, 0)} u(x, t) = 0$
- $\lim_{(x,t) \to (x_0, 0)} u_t(x, t) = 0$

Proof.

$$\begin{aligned} f \in C^2 &\implies v(x, t-s; s) \in C^2(\mathbb{R}^n \times (0, \infty)) \\ &\implies u(x, t) \in C^2(\mathbb{R}^n \times (0, \infty)) \end{aligned}$$

$$u(x, t) = \int_0^t v(x, t-s; s) ds$$

$$\begin{aligned} u_t &= \underbrace{v(x, t-t; t)}_{=0} + \int_0^t v_t(x, t-s; s) ds \\ &= \int_0^t v_t(x, t-s; s) ds \end{aligned}$$

$$\begin{aligned} u_{tt} &= v_t(x, t-t, t) + \int_0^t v_{tt}(x, t-s; s) ds \\ &= v_t(x, 0; t) + \int_0^t v_{tt}(x, t-s; s) ds \\ &= f(x, t) + \int_0^t v_{tt}(x, t-s; s) ds \end{aligned}$$

$$\Delta u = \int_0^t \Delta v(x, t-s; s) ds$$

$$\begin{aligned} \implies u_{tt} - \Delta u &= f(x, t) + \int_0^t v_{tt} - \Delta v ds \\ &= f(x, t) \end{aligned}$$

$$\begin{aligned} u(x, 0) &= \int_0^0 v(0, 0-s; s) ds = 0 \\ u_t(x, 0) &= \int_0^0 v_t(0, 0-s; s) ds = 0 \end{aligned}$$

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last time, $u_t + H(Du, x) = 0$ in $\mathbb{R}^n \times (0, \infty)$

hamiltons equations

$$\begin{cases} \dot{x}(s) = D_p H(p(s), x(s)) \\ \dot{p}(s) = -D_x H(p(s), x(s)) \end{cases}$$

where for $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ smooth, $x(s)$ minimizes

$$I[w(\cdot)] = \int_0^t L(\dot{w}(s), w(s)) ds$$

minimizing I implies the EL equations

$$-\frac{d}{ds}(D_q L(\dot{x}(s), x(s))) + D_x L(\dot{x}(s), x(s)) = 0$$

letting $p(s) := D_q L(\dot{x}(s), x(s))$ and assuming the hypothesis

$$\forall x, p \in \mathbb{R}^n, \ \exists! q \text{ smooth s.t.}$$

$$p = d_q L(q(p, x), x)$$

and we saw that for H defined by

$$H(p, x) := p \cdot q(p, x) - L(q(p, x), x)$$

then $x(s), p(s)$ solve hamiltons equations

legendre transform and hopf lax formula

we now work with $u_t + H(Du) = 0$ in $\mathbb{R}^n \times (0, \infty)$
suppose $L : \mathbb{R}^n \to \mathbb{R}$ satisfies

- $q \to L(q)$ is convex (hence continuous)
- $\lim_{|q| \to \infty} \frac{L(q)}{|q|} = \infty$ superlinear

Definition 0.21. *(legendre transform)*
$L^*(p) := \sup_{q \in \mathbb{R}^n} \{p \cdot q - L(q)\}$

motivation: note that since $\frac{L(q)}{|q|} = \infty$,

$$\sup_q\{p \cdot q - L(q)\} \le \sup_q\{|p||q| - L(q)\}$$

as $|q| \to \infty$, $L(q) \to \infty$ faster than $p \cdot q$
$\implies$ $p \cdot q$ is bounded for $|q|$ large
$\implies$ sup is achieved by a max
i.e. $\exists q^*$ s.t. $L^*(p) = p \cdot q^* - L(q^*)$
thus, $q \to p \cdot q - L(q)$ is maximized at $q = q^*$
$\implies D_q(p \cdot q - L(q))\Big|_{q=q^*} = 0$
$\implies p = D_q L(q^*)$
$\iff \exists q^* = q^*(p)$ s.t. $p = D_q L(q^*(p))$
so $L^*(p) = p \cdot q^*(p) - L(q^*(p)) = H(p)$
so the lagrangian implies the hamiltonian with $H = L^*$

Theorem 0.22. *(convex duality of hamiltonian and lagrangian)*
assume L is convex and superlinear, define $H(p) = L^*(p) = \sup_q\{p \cdot q - L(q)\}$
then

- *$p \to H(p)$ is convex*
- $\lim_{|p|\to\infty} \frac{H(p)}{|p|} = \infty$

also $L = H^*$, so $H = L^*$, $L = H^*$, and we call L, H convex duals of each other.

Proof. claim: $p \to H(p) = L^*(p) = \sup_q\{p \cdot q - L(q)\}$ is convex
by the definition, $\forall \tau \in [0, 1]$, $\forall p, \hat{p} \in \mathbb{R}^n$,

$$H(\tau p + (1-\tau)\hat{p}) = \sup_q\{(\tau p + (1 - \tau\hat{p}) \cdot q - L(q)\}$$

$$= \sup_q\{\tau(p \cdot q - L(q)) + (1-\tau)[\hat{p} \cdot q - L(q)]\}$$

$$\le \tau \sup_q\{p \cdot q - L(q)\} + (1-\tau)\sup\{\hat{p} \cdot q - L(q)\}$$

$$= \tau H(p) + (1-\tau)H(\hat{p})$$

claim: $\liminf_{|p|\to\infty} \frac{H(p)}{|p|} \ge \lambda \ \forall \lambda > 0$
fix $\lambda > 0$, $p \ne 0$.

$$H(p) = \sup_q\{p \cdot q - L(q)\}$$

let $q = \frac{\lambda p}{|p|}$

$$\ge p \cdot \frac{\lambda p}{|p|} - L(\frac{\lambda p}{|p|})$$

$$\geq \lambda|p| - \max_{B(0,\lambda)} L$$

$$\implies \frac{H(p)}{|p|} \geq \lambda - \underbrace{\frac{\max_{B(0,\lambda)} L}{|p|}}_{\to 0 \text{ as } |p|\to\infty}$$

$$\implies \liminf_{|p|\to\infty} \frac{H(p)}{|p|} \geq \lambda$$

claim: $L = H^*$
recall

$$H(p) = L^*(p) = \sup_q \{p \cdot q - L(q)\}$$

$$\implies H(p) \geq p \cdot q - L(q) \;\; \forall p, q \in \mathbb{R}^n$$

$$L(q) \geq p \cdot q - H(p)$$

$$\implies L(q) \geq \sup_p \{p \cdot q - H(p)\} = H^*(q)$$

also,

$$H^*(q) = \sup_p \{p \cdot q - H(p)\}$$

$$= \sup_p \{p \cdot q - \sup_r \{p \cdot r - L(r)\}\}$$

$$= \sup_p \inf_r \{p \cdot (q - r) + L(r)\}$$

since $q \to L(q)$ is convex, $\exists\, s \in \mathbb{R}$ s.t.

$$L(r) \geq L(q) + s \cdot (r - q)$$

$$\geq \sup_p \inf_r \{p \cdot (q - r) + L(q) + s \cdot (r - q)\}$$

let $p = s$

$$\geq \inf_r \{s \cdot (q - r) + L(q) + s \cdot (r - q)\}$$

$$= L(q)$$

■

hopf lax formula

recall we were minimizing $\int_0^t L(\dot{w}(s))ds$ over

$$w \in \mathcal{A} := \{w \in C^1([0,t];\mathbb{R}^n) : w(0) = y, w(t) = x\}$$

fix $x \in \mathbb{R}^n \implies$ "cost to get to x"
what about y? and how do we involve $u(x,0) = g(x)$?
try

$$\int_0^t L(\dot{w}(x))ds + g(w(0))$$

$$\implies u(x,t) := \inf_{w,y}\Big\{\int_0^t L(\dot{w}(s))ds + g(y) : w(0) = y, w(t) = x\Big\} \qquad (*)$$

so now we want to relate this to

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,o) = g(x) & \text{in } \mathbb{R}^n \end{cases}$$

where H is smooth, convex, and superlinear and g is lipschitz continuous i.e.

$$K := \sup_{x,y\in\mathbb{R}^n}\Big\{\frac{|g(x)-g(y)|}{|x-y|}\Big\} < \infty$$

$$\implies |g(x)-g(y)| \leq K|x-y|$$

Theorem 0.23. *(equivalence to hopf lax formula)*
if u is defined by ($\star$)*, then*

$$u(x,t) = \underbrace{\min_y\{tL(\frac{x-y}{t}) + g(y)\}}_{\text{hopf lax formula}}$$

Proof. fix $y \in \mathbb{R}^n$, let $w(x) := y + \frac{s}{t}(x-y)$, note $w(0) = y$ and $w(t) = x$

$$\implies u(x,t) \leq \int_0^t L(\dot{w}(s))ds + g(y)$$

$$= \int_0^t L(\frac{x-y}{t})ds + g(y)$$

$$= tL(\frac{x-y}{t}) + g(y)$$

$$\implies u(x,t) \leq \inf_y\{tL(\frac{x-y}{t}) + g(y)\}$$

also, note that $\forall w(\cdot) \in C^1$ with $w(t) = x$, L is convex, so

$$L(\frac{1}{t}\int_0^t \dot{w}(s)ds) \underbrace{\leq}_{\text{jensen}} \frac{1}{t}\int_0^t L(\dot{w}(s))ds$$

$$L(\frac{1}{t}[x-y]) \leq \frac{1}{t}L(\dot{w}(s))ds$$

$$tL(\frac{x-y}{t}) + g(y) \leq \int_0^t L(\dot{w}(s)ds + g(y)$$

$$\inf_y \left\{tL(\frac{x-y}{t}) + g(y)\right\} \leq \inf_y \left\{\int_0^t L(\dot{w}(s)ds + g(y)\right\}$$

$$\implies \inf_y \left\{tL(\frac{x-y}{t}) + g(y)\right\} \leq u(x,t)$$

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brief aside: viscocity solutions

we introduce a weak notion of solution for

$$(HJ)\begin{cases} u_t + H(Du) = 0 \\ u(x,0) = g(x) \end{cases}$$

we will ask $u : \mathbb{R}^n \times [0,\infty] \to \mathbb{R}$ to be merely continuous.

Definition 0.24. *(viscosity subsolution)*
we say u is a viscosity subsolution of (HJ) if

- $u(x,0) \leq g(x)$ *in* $\mathbb{R}^n$
- $\forall \phi \in C^1(\mathbb{R}^n \times [0,\infty))$, *if $u - \phi$ has a local max at (x_0,t_0)* $\implies$ $\phi_t(x_0,t_0) + H(D\phi(x_0,t_0)) \leq 0$

Remark. observe that if $u \in C^1$ at (x_0,t_0), and

$$u_t(x_0,t_0) + H(Du(x_0,t_0)) = 0$$

then $u-\phi$ has a local max at (x_0,t_0) $\implies$ $u_t(x_0,t_0) = \phi_t(x_0,t_0)$ and $Du(x_0,t_0) = D\phi(x_0,t_0)$

$$\implies \phi_t(x_0,t_0) + H(D\phi(x_0,t_0) = 0$$

i.e. every classical solution is a viscocity solution

it is sufficient to check ϕ touching u from above at (x_0,t_0)

Definition 0.25. *(viscosity supersolution)*
we say u is a viscosity supersolution if

- $u(x,0) \geq g(x)$
- $\forall \phi \in C^1(\mathbb{R}^n \times (0,\infty)))$, *if* $u-\phi$ *has a local min at* (x_0,t_0), *then* $\phi_t + H(D\phi(x_0,t_0)) \geq 0$

it is sufficiant to check ϕ *which touch u from below and check*

$$\phi_t(x_0,t_0) + H(D\phi(x_0,t_0)) \geq 0$$

Definition 0.26. *(viscosity solution)*
we say u is a viscosity solution iff u is both a viscosity subsolution and supersolution.

properties of viscocity solutions

viscosity solutions are unique for (HJ) under very mild assumptions (g is bounded, uniformly continous, and H is lipschitz)

viscocity solutions are the solution you get by studying $\lim_{\epsilon\to 0} u^\epsilon(x,t)$ where

$$u_t^\epsilon - \epsilon\Delta u^\epsilon + H(Du^\epsilon) = 0$$

(it is the limit of a vanishing viscocity problem)

Theorem 0.27. *the hopf lax formula is the unique viscosity solution of* (HJ)

to prove this, we need 1 lemma

Lemma 0.28. *(dynamic programming)*
if

$$u(x,t) = \min_y \left\{ tL(\frac{x-y}{t}) + g(y) \right\}$$

then for all $x \in \mathbb{R}^n$, $0 \leq s \leq t$,

$$u(x,t) = \min_y \left\{ (t-s)L(\frac{x-y}{t-s}) + u(y,s) \right\}$$

to minimize in time t, it is enough to minimize in time s and time $t-s$

Proof. fix $y \in \mathbb{R}^n$, $s \in (0,t)$, let $z \in \mathbb{R}^n$ s.t. $u(y,s) = sL(\frac{y-z}{s}) + g(z)$
note:

$$\frac{x-z}{t} = \frac{x-y}{t} + \frac{s}{t}\frac{y-z}{s}$$

$$= \frac{t-s}{t}\frac{x-y}{t-s} + \frac{s}{t}\frac{y-z}{s}$$

$$= (1-\frac{s}{t})\frac{x-y}{t-s} + \frac{s}{t}\frac{y-z}{s}$$

L is convex

$$L(\frac{x-z}{t}) \le (1-\frac{s}{t})L(\frac{x-y}{t-s}) + \frac{s}{t}L(\frac{y-z}{s})$$

so,

$$u(x,t) \le tL(\frac{x-z}{t}) + g(z)$$

$$\le (t-s)L(\frac{x-y}{t-s}) + \underbrace{sL(\frac{y-z}{s}) + g(z)}_{u(y,s)}$$

$$= (t-s)L(\frac{x-y}{t-s}) + u(y,s)$$

$$\implies u(x,t) \le \min_y \left\{(t-s)L(\frac{x-y}{t-s}) + u(y,s)\right\}$$

let w be s.t.

$$u(x,t) = tL(\frac{x-w}{t}) + g(w)$$

let $y = \frac{s}{t}x + (1-\frac{s}{t})w$

$$\frac{x-y}{t-s} = \frac{1}{t-s}\Big[x - \frac{s}{t}x - \frac{t-s}{t}w\Big]$$

$$= \frac{x}{t} - \frac{w}{t} = \frac{x-w}{t} = \frac{y-w}{s}$$

$$\implies (t-s)L(\frac{x-y}{t-s}) + u(y,s)$$

$$\le (t-s)L(\frac{x-y}{t-s}) + sL(\frac{y-w}{s}) + g(w)$$

$$= (t-s)L(\frac{y-w}{s}) + sL(\frac{y-w}{s} + g(w)$$

$$= tL(\frac{x-w}{t}) + g(w) = u(x,t)$$

$$\implies \min_y \left\{(t-s)L(\frac{x-y}{t-s}) + u(y,s)\right\} \le u(x,t)$$

■

Proof. (of main theorem)

claim: $u(x,t) = \min_y \left\{tL(\frac{x-y}{t}) + g(y)\right\}$ is the unique viscosity solution of (HJ)

as $t \to 0$, $\min_y \left\{tL(\frac{x-y}{t}) + g(y)\right\}$, $|y-x| \le Ct \implies u(x,0) = g(x)$

suppose $\phi \in C^1(\mathbb{R}^n \times (0,\infty))$, and $u - \phi$ has a local max at (x_0, t_0). for any $t \in [0, t_0]$,

$$u(t_0, x_0) = \min_x \{(t_0 - t)L(\frac{x_0 - x}{t_0 - t}) + u(x,t)\}$$

thus $\forall x \in \mathbb{R}^n$,

$$u(x_0, t_0) \leq (t_0 - t)L(\frac{x_0 - x}{t_0 - t}) + u(x, t)$$

■

lecture 11/11

intro to conservation laws

$$(SCL)\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ [u_t + \text{div} \cdot F(u) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x) & \text{in } \mathbb{R} \end{cases}$$

we saw using method of characteristics that u can fail to be C^1 in finite time. our goal is to find a well posed, physical weak solution.

shocks and entropy condition

we saw that viscosity solutions are about using smooth functions to touch at points of nondifferentiability. we could viscosity solutions of scalar conservation laws, but

$$u_t + F(u)_x = u_t + F'(u)u_x$$

is tricky with u dependence.

instead, we note our pde is of the form

$$\partial_t u + \partial_x(F(u)) = 0 \text{ [divergence structure]}$$

suppose u is smooth and solves scl, if $v : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is a smooth, compactly supported test function, then

$$\int_0^\infty \int_{-\infty}^\infty (\partial_t u + \partial_x(F(u))v(x, t)dxdt = 0$$

by ibp, since v is compactly supported, we have

$$\int_{-\infty}^\infty \int_0^\infty -uv_t dtdx + \int_{-\infty}^\infty uv|_{t=0}^{t=\infty} dx - \int_0^\infty \int_{-\infty}^\infty F(u)v_x dxdt = 0$$

$$-\int_0^\infty \int_{-\infty}^\infty uv_t + F(u)v_x dxdt - \int_{-\infty}^\infty g(x)v(x, 0)dx$$

this now makes sense even when u is not differentiable (even u not continuous).

Definition 0.29. *(integral/distributional solution)*

we say $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ is an integral solution of scl provided $\forall v \in C_c^\infty(\mathbb{R}^n \times (0, \infty))$

$$\int_0^\infty \int_{-\infty}^\infty uv_t + F(u)v_x dxdt + \int_{-\infty}^\infty g(x)v(x, 0)dx = 0$$

note: integral solutions need not be continuous, a natural class of functions to consider are the piecewise defined solutions.

simplest case: $V \subset \mathbb{R} \times (0, \infty)$, V divided into V_ℓ and V_r, where u_ℓ, u_r are smooth in the respective domains is an integral solution

suppose v is smooth, $\text{supp}(v) \subset V_\ell$

$$0 = \int_0^\infty uv_t + F(u)v_x dxdt + \underbrace{\int_0^\infty g(x)v(x,0)dx}_{=0}$$

$$= \int\int_{V_\ell} u_\ell v_t + F(u_\ell)v_x dxdt$$

$$= -\int\int_{V_\ell} [(u_\ell)_t + F(u_\ell)_x]v dxdt$$

true for all v smooth, support of v in V_ℓ

$$\implies u_t + F(u)_x = 0 \text{ in } V_\ell$$

similarly, $u_t + F(u)_x = 0$ in V_r

let the curve dividing V be $C = (s(t), t) = (x, t)$, the tangent to C is $(\dot{s}(t), 1)$.

unit tangent is

$$\frac{1}{\sqrt{1 + (\dot{s}(t))^2}}(\dot{s}(t), 1)$$

unit normal is

$$\frac{1}{\sqrt{1 + (\dot{s}(t))^2}}(1, -\dot{s}(t))$$

$$F(u_\ell) - F(u_r) = \dot{s}(t)(u_\ell - u_r)$$

Definition 0.30. *(rankine hogonoit condition)*

a piecewise smooth function satisfies the rankine hugoniot condition (r-h condition) if

$$F(u_\ell) - F(u_r) = \dot{s}(t)(u_\ell - u_r) \text{ on } C$$

we have shown every integral solution must satisfy the r-h condition (should also be sufficient)

lecture 11/18

last time,

$$(SCL)\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = g(x) & \text{in } \mathbb{R} \end{cases}$$

lax-oleinik formula: $L = F^*$

$$u(x,t) = \frac{\partial}{\partial x}\Big[\min_y \Big\{tL(\frac{x-y}{t}) + h(y)\Big\}\Big]$$

$$= G(\frac{x-y(x,t)}{t}) \qquad (star)$$

a.e. where $G = (F')^{-1}$
and

$$\min_y \Big\{tL(\frac{x-y}{t}) + h(y)\Big\} = tL(\frac{x-y(x,t)}{t}) + h(y(x,t))$$

$h(x) = \int_{-\infty}^{x} g(y)dy$ and $x \to y(x,t)$ is nondecreasing
we will now show $u(x,t)$ is an integral solution of (SCL).

Theorem 0.31. *u defined by* $(\star)$ *is an integral solution of* (SCL)

Proof. recall $w(x,t) = \min_y \Big\{tL(\frac{x-y}{t}) + h(y)\Big\}$, it is shown in hw that w is lipschitz and differentiable a.e., and at points of differentiability,

$$w_t + F(w_x) = 0 \text{ in } \mathbb{R}\times(0,\infty)$$

$$w(x,0) = h \text{ in } \mathbb{R}$$

let $v \in C_c^\infty(\mathbb{R}\times(0,\infty))$,

$$\int_0^\infty \int_{-\infty}^\infty \underbrace{[w_t + F(w_x)]}_{=0 \text{ a.e.}} v_x(x,t)dxdt = 0 \qquad (\square)$$

and moreover, the fact that w is differentiable a.e. means we can integrate by parts

$$\int_0^i nfty \int_{-\infty}^\infty w_t v_x dxdt = -\int_0^\infty \int_{-\infty}^\infty w v_{xt} dxdt - \int_{-\infty}^\infty w v_x dx|_{t=0}$$

$$= \int_0^\infty \int_{-\infty}^\infty w_x v_t dxdt + \int_{-\infty}^\infty w_x v dx|_{t=0}$$

note $w(x,0) = h(x) = \int_{-\infty}^{x} g(y)dy$
$w_x(x,0) = g(x)$
returning back to $(\square)$

$$0 = \int_0^\infty \int_{-\infty}^\infty \underbrace{[w_t + F(w_x)]}_{=0 \text{ a.e.}} v_x(x,t)dxdt$$

$$= \int_0^\infty \int_{-\infty}^\infty w_x v_t dxdt + \int_{-\infty}^\infty g(x)v(x,0)dx + \int_0^\infty \int_{-\infty}^\infty F(w_x)v_x dxdt$$

letting $u(x,t) = w_x(x,t)$ a.e.

$$= \int_0^\infty \int_{-\infty}^\infty uv_t + F(u)v_x dxdt + \int_{-\infty}^\infty g(x)v(x,0)dx$$

so lax olenik is an integral solution. to check if it is an entropy solution, we need a general entropy condition.

Definition 0.32. *(oleinik entropy condition)*
u satisfies the oleinik entropy condition if $\exists C > 0$ *s.t. for a.e.* $x, z \in \mathbb{R}$, $t \geq 0, z > 0$

$$u(x+z,t) - u(x,t) \leq \frac{c}{t} z \tag{$\star$}$$

$$[u(x+z,t) - \frac{c}{t}(x+z) \leq u(x,t) - \frac{c}{t}x]$$

Remark. this implies $x \mapsto u(x,t) - \frac{c}{t}x$ is nonincreasing, which implies left and right hand limits u_ℓ, u_r where

$$u_\ell(x,t) \geq u_r(x,t)$$

(this was the original condition)

Lemma 0.33. *the lax oleinik formula satisfies the oleinik entropy condition*

Proof. we will assume $G = (F')^{-1}$ is lipschitz
since G is lipschitz and $x \mapsto y(x,t)$ is nondecreasing,

$$u(x,t) = G(\frac{x - y(x,t)}{t}) \geq G(\frac{x - y(x+z,t)}{t})$$

for $z > 0$

$$\geq G(\frac{x + z - y(x+z,t)}{t}) - lip(G)\frac{z}{t}$$

$$= u(x+z,t) - lip(G)\frac{z}{t}$$

$$\implies u(x+z,t) - u(x,t) \leq lip(G)\frac{z}{t}$$

■

so lax-olenik formula is the unique entropy solution of (SCL)

holder spaces

let $U \subset \mathbb{R}^n$, $\gamma \in (0,1]$.

recall that $u : U \to \mathbb{R}$ is lipschitz continuous if $\exists\, c > 0$ s.t. $|u(x) - u(y)| \le c|x-y|$

we need

$$c \ge \underbrace{\sup_{x,y\in U, x\neq y} \frac{|u(x)-u(y)|}{|x-y|}}_{\text{needs to be finite}}$$

now,

∎

Definition 0.34. *(holder continuous)*

$u : U \to \mathbb{R}$ is holder continuous with exponent $\gamma \in (0,1]$ if $\exists\, c > 0$ s.t.
$|u(x) - u(y)| \le c|x-y|^\gamma \ \forall x, y \in U$

Definition 0.35. *(holder seminorm/norm)*

- *if $u : U \to \mathbb{R}$ is bounded and continuous,*

$$\|u\|_{C(\overline{U})} := \sup_{x\in U} |u(x)|$$

- *the γ-holder seminorm of u is*

$$[u]_{C^{0,\gamma}(\overline{U})} := \sup_{x,y\in U, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$

the γ-holder norm is then defined by

$$\|u\|_{C^{0,\gamma}(\overline{U})} := \|u\|_{C(\overline{U})} + [u]_{C^{0,\gamma}(\overline{U})}$$

Definition 0.36. *(higher order holder spaces)*

$C^{k,\gamma}(\overline{U})$ consists of all $u \in C^k(\overline{U})$ s.t.

$$\|u\|_{C^{k,\gamma}(\overline{U})} := \sum_{|\alpha|\le k} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})} < \infty$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$

Theorem 0.37. *$C^{k,\gamma}(\overline{U})$ is a banach space*

Proof. hw ∎

so $u_j \to u \in C^{k,\gamma}(\overline{U})$ iff $\|u_j - u\|_{C^{k,\gamma}(\overline{U})} \to 0$ as $j \to \infty$

Remark. holder continuous $\implies$ uniformly continuous

sobolev spaces

weak derivatives

recall that $\phi \in C_c^\infty(U)$ is what we call a smooth, compactly supported test function.
motivation: integration by parts

$$\int_U u\phi_{x_i}dx = -\int_U u_{x_i}\phi dx$$

Definition 0.38. *(weak derivative)*
let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, $D^\alpha u = \partial_{x_i}^{\alpha_1}\partial_{x_2}^{\alpha_2}\ldots\partial_{x_n}^{\alpha_n}u$
suppose $u, v \in L^1_{loc}$, *then* $v = D^\alpha u$ *if* $\forall \phi \in C_c^\infty(U)$,

$$\int_U uD^\alpha \phi dx = (-1)^{|\alpha|}\int_U v\phi dx$$

Remark. we need $D^\alpha u = v$ to be a function, these are unique up to a set of measure 0.

example. $n = 1, U = (0, 2)$

$$u(x) = \begin{cases} x & 0 < x \leq 1 \\ 1 & 1 \leq x < 2 \end{cases}$$

$$v(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & 1 \leq x < 2 \end{cases}$$

claim: $v = u'$ in weak sense
let $\phi \in C_c^\infty((0, 2))$

$$\int_0^2 u\phi' dx = \int_0^1 x\phi'(x)dx + \int_1^2 \phi'(x)dx$$

$$= -\int_0^1 \phi(x)dx + \phi(1) - \phi(1) = -\int_0^1 \phi(x)dx$$

$$= -\int_0^2 \phi(x)v(x)dx$$

def of sobolev space

Definition 0.39. *(sobolev space)*
the sobolev space

$$W^{k,p}(U) := \{u : U \to \mathbb{R} \ s.t. \ \forall |\alpha| \leq k, D^\alpha u \in L^p(U)\}$$

$$||u||_{W^{k,p}}(U) := \begin{cases} \sum_{|\alpha|\leq k}\left(\int_U |D^\alpha u|^p dx\right)^{\frac{1}{p}} & p < \infty \\ \sum_{|\alpha|\leq k} ess\ sup_U |D^\alpha u| & p = \infty \end{cases}$$

we will call $H^k(U) = W^{k,2}(U)$, *derivatives belong to* $L^2(U)$

Definition 0.40. *(convergence)*

- $u_m \to u$ *in* $W^{k,p}(U)$ *if* $\lim_{m\to 0} \|u_m - u\|_{W^{k,p}(U)} = 0$
- $u_m \to u$ *in* $W^{k,p}_{loc}(U)$ *if* $u_m \to u$ *in* $W^{k,p}(V)\ \forall V \subset\subset U$

Definition 0.41. *($W_0^{k,p}(U)$)*
we denote $W_0^{k,p}(U)$ *to be the closure of* $C_c^\infty(U)$ *in the* $W^{k,p}$ *norm.*
$u \in W_0^{k,p}(U)$ *iff* $\exists \{u_m\} \subset C_c^\infty$ *s.t.* $u_m \to u$ *in* $W^{k,p}(U)$ *iff* $D^\alpha u = 0$ *on* ∂U, $|\alpha| \le k-1$

Remark. $H_0^k(U) = W_0^{k,2}(U)$.

properties of weak derivatives

if $u, w \in W^{k,p}(U)$,

- composition $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$
- linear combinations $\lambda u + \mu v \in W^{k,p}(U)$, $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$ for all $\lambda, \mu \in \mathbb{R}$
- if $V \subset U$, then $u, v \in W^{k,p}(V)$
- if $\phi \in C_c^\infty$, then $\phi u \in W^{k,p}(U)$ and we have a product rule
  $D(\phi u) = D\phi u + \phi D u$

lecture 11/23

last time, we saw for $U \subset \mathbb{R}^n$ open, bounded,

$$W^{k,p}(U) := \{u : U \to \mathbb{R} \text{ s.t. } \forall |\alpha| \le k, D^\alpha u \in L^p(U)\}$$

$$\|u\|_{W^{k,p}}(U) := \begin{cases} \sum_{|\alpha|\le k} \left(\int_U |D^\alpha u|^p dx \right)^{\frac{1}{p}} & p < \infty \\ \sum_{|\alpha|\le k} \text{ess sup}_U |D^\alpha u| & p = \infty \end{cases}$$

$$W_0^{k,p}(U) := \text{ closure of } C_c^\infty(U)$$

in the $W^{k,p}$ norm.

Theorem 0.42. *($W^{k,p}(U)$ is a banach space)*
$\forall k$, $p \in [1, \infty]$, $W^{k,p}(U)$ *is a banach space.*

Proof. note $\forall \lambda \in \mathbb{R}$, we have

$$\|\lambda u\|_{W^{k,p}(U)} = |\lambda| \|u\|_{W^{k,p}(U)}$$

and

$$\|u\|_W^{k,p}(U) = 0 \implies \|u\|_{L^p} = 0 \implies u \equiv 0 \text{ (a.e.)}$$

assume $u, v \in W^{k,p}(U)$. then

$$\|u+v\|_W^{k,p}(U) = \Big(\sum_{|\alpha|\le k} \|D^\alpha u + D^\alpha v\|_{L^p}^p\Big)^{\frac{1}{p}}$$

$$\le \left(\sum_{|\alpha|\le k} \Big(\|D^\alpha u\|_{L^p} + \|D^\alpha v\|_{L^p}\Big)^p\right)^{\frac{1}{p}}$$

$$\le \Big(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p}^p\Big)^{\frac{1}{p}} + \Big(\sum_{|\alpha|\le k} \|D^\alpha v\|_{L^p}^p\Big)^{\frac{1}{p}}$$

$$= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}$$

to show it is complete, suppose $\{u_m\}_{m=1}^\infty$ is a cauchy sequence.
$\implies \{D^\alpha u_m\}_{m=1}^\infty$ is cauchy in L^p
L^p is complete

$$D^\alpha u_m \to u_\alpha \text{ in } L^p(U)$$

$$u_m \to u \text{ in } L^p(U)$$

claim: $u \in W^{k,p}(U)$ with $D^\alpha u = u_\alpha$
let $\phi \in C_c^\infty(U)$. then

$$\int_U u D^\alpha \phi dx = \lim_{m\to\infty} \int_U u_m D^\alpha \phi dx$$

$$= \lim_{m\to\infty} (-1)^{|\alpha|} \int_U D^\alpha u_m \phi dx$$

$$= (-1)^{|\alpha|} \int_U u_\alpha \phi dx$$

$$\implies u \in W^{k,p}, D^\alpha u_m \to D^\alpha u \text{ in } L^p$$

$$\implies u_m \to u \text{ in } W^{k,p}(U)$$

■

lecture 12/2

maximum principles

we move to max principles for nondivergence form equations

$$Lu = -\sum_{i,j} a_{ij}(x)u_{x_i x_j} + \sum_i b_i u_{x_i} + cu(x)$$

- $a_{ij}(x)$ uniform elliptic, i.e. $a_{ij}(x) \geq \theta \mathrm{id}$
- $a_{ij}(x)$ symmetric
- a_{ij}, b_i, c are continuous

we will work with $w \in C^2(U) \cap C(\overline{U})$

weak max principle

$U \subset \mathbb{R}^n$ is open and bounded

Theorem 0.43. *(weak max principle (WMP))*
assume $u \in C^2(U) \cap C(\overline{U})$ *and* $c \equiv 0$ *in* U*,*

- *if* $Lu \leq 0$ *in* U*, then* $\max_{\overline{U}} u = \max_{\partial U} u$
- *if* $Lu \geq 0$ *in* U*, then* $\min_{\overline{U}} u = \min_{\partial U} u$

Proof. first assume $Lu < 0$ in U and $\exists x_0 \in U$ s.t. $u(x_0) = \max_{\overline{U}} u$

$$\implies Du(x_0) = 0,\ D^2 u(x_0) \leq 0$$

since $A := a_{ij}(x_0)$ is symmetric, positive definite, it is orthogonally diagonalizable, i.e. there is a basis of orthogonal eigenvectors

let $y := x_0 + O(x - x_0)$, $O^T(y - x_0) = (x - x_0)$

$$\implies u_{x_i} = \sum_k u_{y_k} \frac{\partial y_k}{\partial x_i} = \sum_k u_{y_k} O_{ki}$$

$$u_{x_i x_j} = \sum_{k,\ell} u_{y_k y_\ell} O_{ki} O_{\ell j}$$

$$\implies \sum_{i,j} a_{ij} u_{x_i x_j} = \sum_{k,\ell} \sum_{i,j} \underbrace{O_{ki} a_{ij} O_{\ell j}}_{O^T AO = D} u_{y_k y_\ell}$$

$$= \sum_k d_k u_{y_k y_k}$$

since $d_k \geq \theta > 0$ by uniform ellip, and $u_{y_k y_k} \leq 0$,

$$\leq 0$$

so

$$Lu = -\underbrace{\sum_{i,j} a_{ij}(x_0)u_{x_ix_j}}_{\geq 0} + \underbrace{\sum_i b_i(x_0)u_{x_i}}_{u_{x_i}(x_0)=0}$$

$$\geq 0$$

but this contradicts $Lu < 0$ in $U \implies x_0 \in \partial U$, $\max_{\overline{U}} u = \max_{\partial U} u$
if $Lu \leq 0$, write

$$u^\epsilon(x) := u(x) + \epsilon e^{\lambda x}$$

for $\lambda > 0$ to be chosen.
by linearity, $Lu^\epsilon = Lu + \epsilon L e^{\lambda x}$
tbc ■

Theorem 0.44. *(wmp, $c \geq 0$)*
let $u \in C^2(U) \cap C(\overline{U})$, and $c \geq 0$ in U,

- *if $Lu \leq 0$ in U, then* $\max_{\overline{U}} u \leq \max_{\partial U} u^+$
- *if $Lu \geq 0$ in U, then* $\min_{\overline{U}} u \leq -\max_{\partial U} u^-$

Remark. so if $Lu = 0$ in U, then

$$\max_{\overline{U}} |u| = \max_{\partial U} |u|$$

Proof. let $Lu \leq 0$ and set $V := \{x \in U : u(x) > 0\}$ (open)
let $Ku := Lu - cu \leq 0 - cu \leq 0$ in V
so Ku is a uniform elliptic operator with no 0th order term, V open

$$\max_{\overline{V}} u = \max_{\partial V} u \leq \max_{\partial U} u^+$$

this is equal to $\max_{\overline{U}} u$ if V is nonempty
if V is empty, then $u(x) \leq 0$ everywhere

$$\implies \max_{\overline{U}} u \leq 0 = \max_{\partial U} u^+$$

supersolution is the same argument, u is a supersolution implies $-u$ is a subsolution,

$$\max_{\overline{U}} -u \leq \max_{\partial U} (-u)^+$$

$$-\min_{\overline{U}} u \leq \max_{\partial u} u^-$$

$$\min_{\overline{U}} u \leq -\max_{\partial u} u^-$$

Remark. we get comparison principle for free

Remark. what if $u \notin C^2(U) \cap C^{\overline{U}}$?
note both arguments only relied on $Du(x_0) = 0$ and $D^2u(x_0) \leq 0$.

Definition 0.45. *(viscocity solutions for 2nd order equations)*
consider Lu as above, $c \geq 0$*,*
u is a viscosity subsolution if $\forall \phi \in C^2(U)$*,* $u - \phi$ *has a local max on* x_0 *implies*

$$L\phi(x_0) \leq 0$$

u is a viscosity supersolution if $\forall \phi \in C^2(U)$*,* $u - \phi$ *has a local min on* x_0 *implies*

$$L\phi(x_0) \geq 0$$

a viscosity solution is a viscozity subsoltion and a viscosity supersolution.

so now, if $Lu < 0$ in the viscosity sense, u has a local max at $x_0 \implies u - \phi$ has a local max for $\phi(x) \equiv 0$

■